# Lattice Packings of Face-Embedded Archimedean Solids

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**Abstract**

Each Archimedean solid can be face-embedded into a regular Tetrahedron [4]. This condition extends, in some cases, such that pairs of Archimedean solids face-embed into each other. One special case is the truncated Octahedron ($tO$). $tO$ classically tessellates $\mathbb{R}^3$, and each other Archimedean solid can be face-embedded into $tO$. Presently we discuss how this embedding can produce other lattice packings of $\mathbb{R}^3$.

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## 1 The Archimedean Solids

Firstly, we introduce the Archimedean solids. Below we list a table providing the names, notation, and number of face types for each solid.

A note on notation, here we are adopting a modified version of Conway's polyhedron notation [2]. The lowercase prefix denotes the operation that is applied to the seed solid represented by uppercase the letters. $t, e, s$ represent truncation, expansion, and snubification respectively. $T, C, O, D, I$ all represent the obvious Platonic solid with matching first letter. For the face types $\{n\}$ represents a regular polygon face with $n$ sides e.g. $\{3\}$ is an equilateral triangle.

| **Name** | **Notation** | **Face Types** | | |
|---|---|---|---|---|
| truncated Tetrahedron | $tT$ | {3}:4 | {6}:4 | |
| Cuboctahedron | $CO$ | {3}:8 | {4}:6 | |
| truncated Cube | $tC$ | {3}:8 | {8}:6 | |
| truncated Octahedron | $tO$ | {4}:6 | {6}:8 | |
| Rhombicuboctahedron | $eO$ | {3}:8 | {4}:18 | |
| truncated Cuboctahedron | $tCO$ | {4}:12 | {6}:8 | {8}:6 |
| Snub Cube | $sC$ | {3}:32 | {4}:6 | |
| Icosidodecahedron | $ID$ | {3}:20 | {10}:12 | |
| truncated Dodecahedron | $tD$ | {3}:20 | {12}:12 | |
| truncated Icosahedron | $tI$ | {6}:20 | {10}:12 | |
| Rhombicosidodecahedron | $eI$ | {3}:20 | {4}:30 | {10}:12 |
| truncated Icosidodecahedron | $tID$ | {4}:30 | {6}:20 | {12}:12 |
| Snub dodecahedron | $sD$ | {3}:80 | {10}:12 | |

Table 1: Archimedean Solids

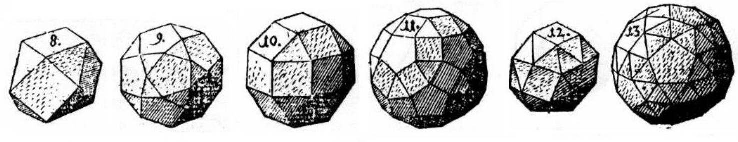


Figure 1: Kepler's Drawing of 8. CO, 9. tD, 10. eO, 11. tID, 12. sC, 13. sD [3]

## 2 Lattice Packings

A (Bravais) lattice packing $L$ is one in which the centroids of the non-overlapping particles are located at the points of $L$, each oriented in the same direction. $\mathbb{R}^3$ can then be geometrically divided into identical regions $F$ called fundamental cells, each of which contains just the centroid of one particle. A periodic packing of particles $S$ is obtained by placing a fixed non-overlapping configuration of $N$ particles (where $N \geq 1$) with arbitrary orientations in each fundamental cell of $L$. Thus, the packing is still periodic under translations by $L$, but the $N$ particles can occur anywhere in the chosen cell subject to the non-overlap condition. The density of a periodic packing is $\frac{N\cdot(\text{vol}(S))}{\text{vol}(F)}$.

Many have observed that $tO$ tessellates $\mathbb{R}^3$, as the bitruncated cubic honeycomb [1]. The centroid of each $tO$ forms a standard body-centered cubic (bcc) lattice. By the main theorem of [4] we know that each of the Archimedean solids is 4 face-embeddable in $T$. There is a natural extension were we can consider if two Archimedean solids are n-face-embeddable in each other. Shown below 2 is a $tI$ 8 face-embedded in a $tO$. For each pair of Archimedean solids mark the cell gray in the following table 2 if they are not strictly contained within one another. Otherwise there is an 8-face-embedding of the pair.

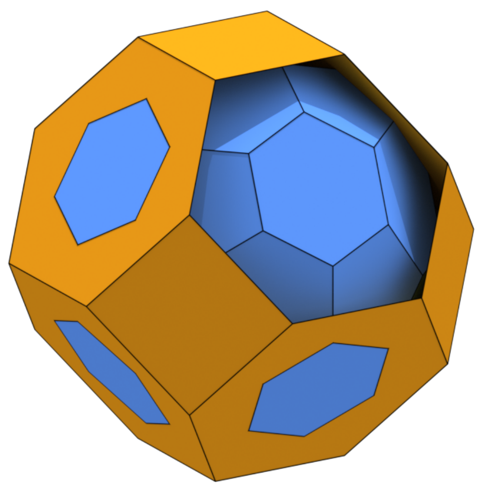

Figure 2: A truncated Icosahedron 4 face-embedded in a truncated Octahedron

| | $tT$ | $tO$ | $tI$ | $tCO$ | $sC$ | $CO$ | $eO$ | $sD$ | $tID$ | $eI$ | $IO$ | $tD$ | $tC$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $tT$ | 1.000 | 0.522 | 0.402 | 0.387 | 0.374 | 0.367 | 0.357 | 0.356 | 0.355 | 0.351 | 0.339 | 0.292 | 0.242 |
| $tO$ | | 1.000 | 0.770 | 0.743 | 0.717 | 0.703 | 0.684 | 0.682 | 0.680 | 0.673 | 0.065 | 0.559 | 0.464 |
| $tI$ | | | 1.000 | 0.964 | 0.931 | 0.913 | 0.888 | 0.885 | 0.883 | 0.874 | 0.845 | 0.725 | 0.602 |
| $tCO$ | | | | 1.000 | 0.966 | 0.947 | 0.921 | 0.918 | 0.916 | 0.907 | 0.876 | 0.752 | 0.624 |
| $sC$ | | | | | 1.000 | 0.980 | 0.953 | 0.950 | 0.948 | 0.939 | 0.907 | 0.779 | 0.646 |
| $CO$ | | | | | | 1.000 | 0.973 | 0.969 | 0.967 | 0.958 | 0.925 | 0.795 | 0.659 |
| $eO$ | | | | | | | 1.000 | 0.997 | 0.995 | 0.985 | 0.951 | 0.817 | 0.678 |
| $sD$ | | | | | | | | 1.000 | 0.998 | 0.988 | 0.954 | 0.820 | 0.680 |
| $tID$ | | | | | | | | | 1.000 | 0.990 | 0.956 | 0.822 | 0.682 |
| $eI$ | | | | | | | | | | 1.000 | 0.966 | 0.830 | 0.689 |
| $ID$ | | | | | | | | | | | 1.000 | 0.859 | 0.713 |
| $tD$ | | | | | | | | | | | | 1.000 | 0.830 |
| $tC$ | | | | | | | | | | | | | 1.000 |

Table 2: Ratio of Volumes Between Archimedean Solids

The value in a given cell is the volume of the solid labeled in the column divided by the solid labeled in the row. For example in row two column 3 the value $0.770 = \frac{\mathrm{vol}(tI)}{\mathrm{vol}(tO)}$ is the ratio for the pair shown in 2. It is important to note that every solid is scaled such that it can be 4 face-embedded into the same $T$. In more technical terms this means that they are scaled such that the insphere, to the 3-fold face of smallest order, all have the same radius. We now note that every Archimedean solid is 4 face-embeddable in $tO$ (see row 2). This means that each of the values in row 2 is exactly the density of a bcc lattice packing formed by that solid.

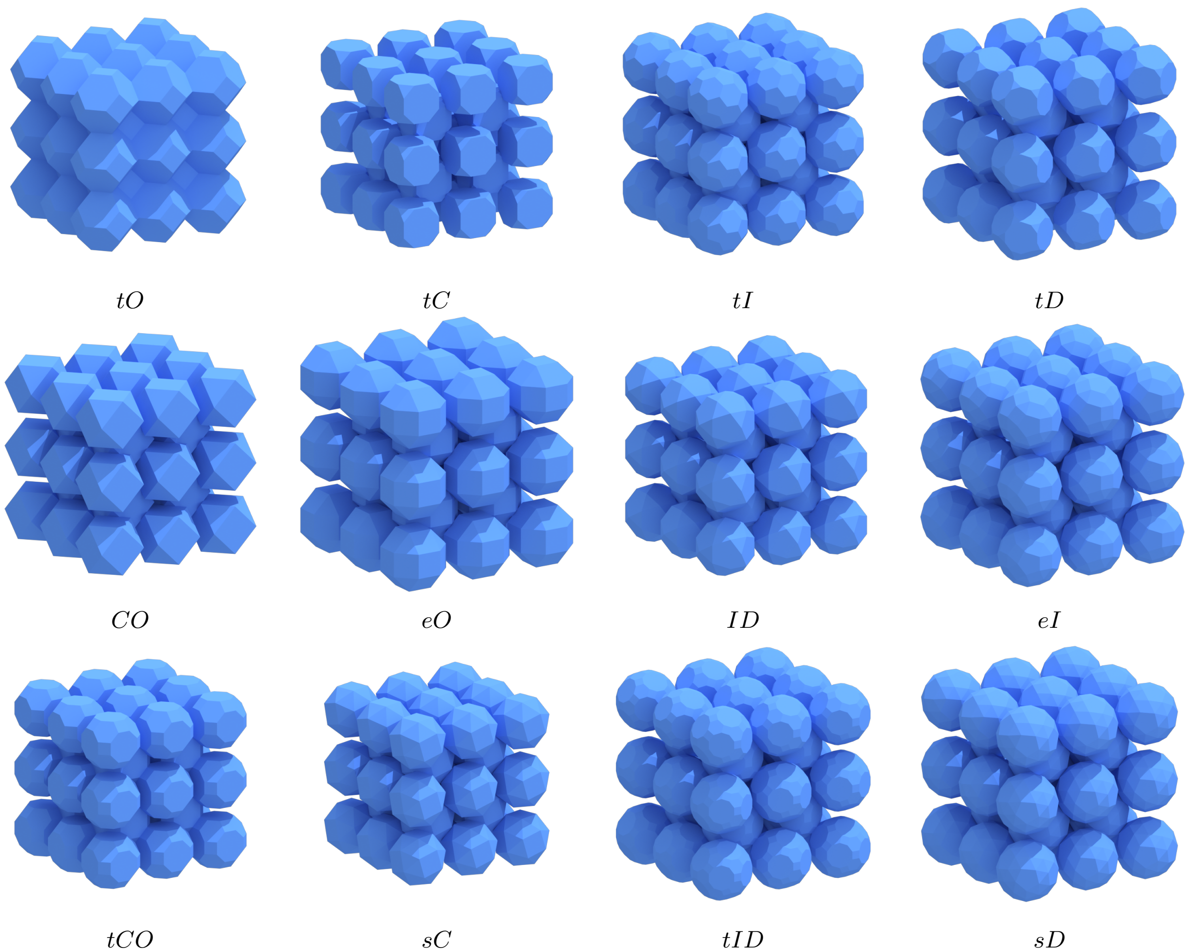


Figure 3: Body-Centered Cubic Lattice Packings of Archimedean Solids

# References


[1] J.H Conway, H. Burgiel, and C. Goodman-Strauss. *The symmetries of things*. A K Peters/CRC Press, 2016.

[2] G. Hart. http://www.georgehart.com/virtual-polyhedra/conway_notation.html.

[3] J. Kepler, E.J. Aiton, A.M. Duncan, and J.V. Field. *The Harmony of the World*. American Philosophical Society: Memoirs of the American Philosophical Society. American Philosophical Society, 1997.

[4] Tommy Murphy and David Weed. Face embeddings of archimedean solids, 2024.

# A Lattice Generating Code

Here we provide the code used to generate the images in 3. This is a Grasshopper® script for the software Rhino®. Rhino is primarily used for 3D modeling in fields such as architecture, however Grasshopper provides many tools useful for computational geometry.

The primary action of this script is to move copies of out selected solid (truncated Cube, $tC$) outwards in the direction of the faces of an Octahedron ($O$). The "Move Away From" takes in the face meshes of $O$, and the solid $tC$ then shifts $tC$ by a distance that we define. To find the appropriate distance we use the "Face Normals" block to pick out the center of each face of $tC$. We calculate the distance of each of these then we use the "Number Slider" to pick the appropriate distance. In geometric terms we are doubling the radius of the insphere of a specific face type. Then after we move away each solid we repeat this transformation to get one section of the lattice. Finally, the material and preview blocks allow us to yield the nicely rendered images.

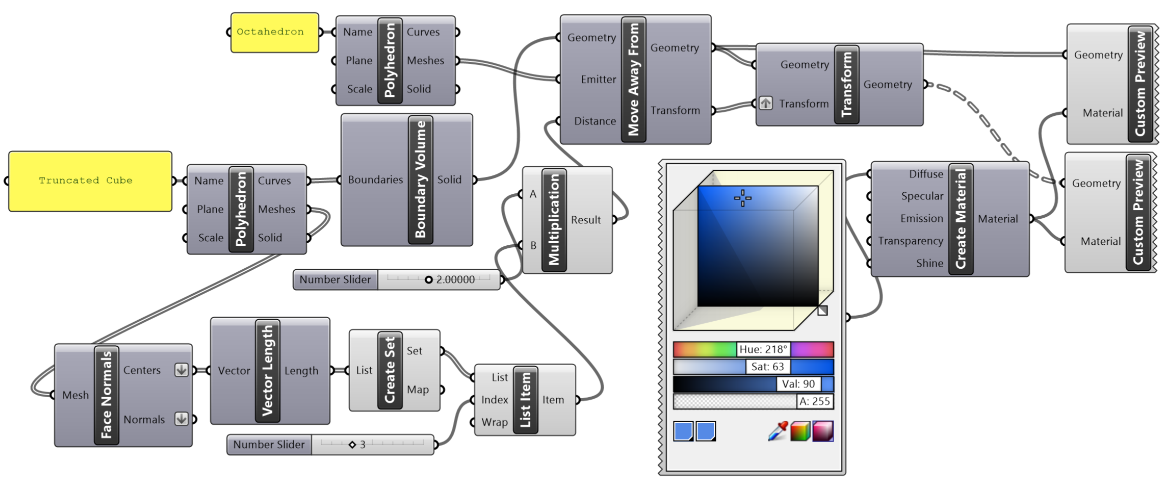


Figure 4: Grasshopper Script