MAXWELL'S CONJECTURE ON FOUR COLLINEAR POINT CHARGES

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Abstract. We study an old conjecture of Maxwell concerning the critical points of an electrostatic potential with finitely many point charges under the condition that the point charges are collinear. If the outermost charge has different sign to all others, we show there are finitely many critical points. In the case where there are four point charges, we investigate the conjecture in a special case.

1. Introduction

1.1. **Background.** A vector field $\mathbb{E}$ on a domain $\Omega \subseteq \mathbb{R}^3$ is said to be conservative if $\mathbb{E} = \nabla\phi$ for some function $\phi : \Omega \to \mathbb{R}$. These occur throughout physics and engineering. A point charge is defined as a charge $q \in \mathbb{R}\backslash\{0\}$ located at a point $\mathbf{x} \in \mathbb{R}^{\mathbf{3}}$. Given charges q_i located at distinct $\mathbf{x}_i$, $i = 1, \dots, n$, we define $\Omega := \mathbb{R}^3\backslash\{\mathbf{x}_i,\ i = 1, \dots, n\}$ and the electrostatic (or Newtonian) potential $\phi : \Omega \to \mathbb{R}$ via

$$\phi(\mathbf{x}) := \sum_{i=1}^{n} \frac{q_i}{\|\mathbf{x} - \mathbf{x}_i\|}.$$

Its gradient $\mathbb{E} = \nabla\phi$ is the electrostatic field generated by the point charges.

Example 1.1. *A standard example is the vector field*

$$\mathbb{G}(\boldsymbol{x}) = \mathbb{G}(x, y, z) = \left\langle \frac{-x}{(x^2+y^2+z^2)^{3/2}}, \frac{-y}{(x^2+y^2+z^2)^{3/2}}, \frac{-z}{(x^2+y^2+z^2)^{3/2}} \right\rangle,$$

which is conservative as $\mathbb{G} = \nabla\phi$, where $\phi : \mathbb{R}^3\backslash\{\mathbf{0}\}$ is the potential energy defined as

$$\phi(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}.$$

This paper is concerned with the following classical question, raised by Maxwell [3]. If there are finitely many distinct point charges, how many points are there in Ω where the vector field vanishes? At such points $\mathbf{x}$ no force is felt by an observer, and we have $\mathbb{E}(\mathbf{x}) = \nabla\phi(\mathbf{x}) = 0$. Thus $\mathbf{x}$ is a critical point of ϕ. Maxwell conjectured that if there are n point charges the number of critical points, if finite, is at most $(n-1)^2$.

This is still largely open. It is not even known in general under what conditions the number of critical points will be finite. Maxwell's conjecture has a long history, and generalizes work of Gauß on electrostatic potentials in $\mathbb{R}^2$. Recent progress on related problems is outlined in [2] and [5]. In $\mathbb{R}^3$, most work to date has focused on the case where the potential is a Morse function, which means all the critical points are nondegenerate. In fact, in Section 32 of [4] Cairns and Morse study this situation as an application of Morse Theory. They prove a Theorem which applies when all the charges lie on a line (or more generally form a so-called minimal configuration) which realizes a lower bound on the number of critical points via the Morse Inequalities.

In [6], Tsai proves the conjecture for the case where $n = 3$ under some assumptions on the charges. Tsai proves that there are either two, three or four critical points if they are isolated, and describes exactly how all these cases occur. For the collinear case (i.e. all charges are on the same line in $\mathbb{R}^3$), he proves

Theorem 1.2. *Assume three point charges are located at (1,0,0),(-1,0,0) and $(u, v, 0)$ with charges $s^3, k^3, 1$ respectively. For $s = k = 1$, there are two critical points for ϕ if $v = 0$, $u \neq \pm 1$. For $s = k = -1$, there are two critical points for ϕ if $v = 0$, $\|u\| > 1$, and there is a circle of critical points for ϕ if $v = 0$ and $|u| < 1$.*

In the case where $v = 0$ and $|u| < 1$ this means there are infinitely many critical points. This result establishes the conjecture of Maxwell for the case of three collinear point charges with equal charges. In this paper we extend Tsai's work to consider the case where there are four or more collinear point charges, under the assumption one of the outermost point charges has a different sign to all the others. The assumption all point charges are collinear enable us to reduce this to a one-dimensional problem which is possible to solve using techniques from calculus and classical results from the study of roots of real polynomials. As pointed out to us by G. Jennings, S. Raianu and W. Horn, in fact there is an easy calculus argument (see Proposition 2.4) which establishes Maxwell's conjecture in the collinear case when the charges are all positive. This also covers the first case of Tsai's theorem with $n = 3$ and makes his theorem redundant in this case.

The main results are as follows.

Theorem A. *If there are n collinear point charges so that the outermost charge has opposite sign to all others, then ϕ has at most $2(n^2 - 1)$ critical points.*

In particular, this number is finite, but this is not enough to establish Maxwell's conjecture. When $n = 4$, this shows the number of critical points is at most 30. Our main theorem improves this significantly in a special case.

Theorem B. *Suppose there are four collinear point charges such that $-q_1 = q_2 = q_3 = q_4$. Then ϕ has at most twelve critical points.*

This is not optimal: the arguments presented here can be adapted to various other configurations of charges but the general conjecture remains intractable even in the collinear case.When $n = 4$, we have $(4 - 1)^2 = 9$ and so Theorem B does not establish Maxwell's conjecture in this case. If one additionally positions the charges in special locations along the line it is possible to verify the conjecture in many special cases. The reader might note that a natural situation to investigate include when $n = 3$ and the collinear charges satisfy the corresponding assumption $-q_1 = q_2 = q_3$. Then the problem reduces to finding roots of quartics, which of course can be analysed exactly since there is an explicit formula for finding the roots of a quartic. Our goal in this work however was to investigate a test case where the degree of the polynomials is too large to directly find the roots.

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2. Preliminaries

2.1. **Initial Setup and Generalities.** We briefly review some material from multivariable calculus which will be required. Take a function $F : \mathbb{R}^n \to \mathbb{R}^m$, written in components as

$$F(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), f_2(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n))$$

for $f_i : \mathbb{R}^n \to \mathbb{R}$. Then the Jacobian is the $(m \times n)$ matrix

$$dF = \left(\frac{\partial f_i}{\partial x_j}\right)_{ij}$$

where $1 \leq i \leq m$, $1 \leq j \leq n$. Note that if $m = 1$, the Jacobian is a $1 \times n$ row matrix which is exactly ∇F.

Definition 2.1. A map $F : \mathbb{R}^3 \to \mathbb{R}^3$ is said to be a *change of coordinates* if the Jacobian dF has non-zero determinant.

Proposition 2.2. *Critical points are invariant under changes of coordinates.*

Proof. Letting $\mathbf{x} = (x, y, z)$ be the original coordinate system, and $\tilde{\mathbf{x}} = F(\mathbf{x})$ be the new coordinate system. Then $\phi(\tilde{x}) := \phi \circ F^{-1}(\tilde{x})$, so the chain rule states that

$$d\phi(\tilde{\mathbf{x}}) = (d\phi(\mathbf{x}))(dF^{-1})$$

Since F is a change of coordinates, $(dF^{-1}) = (dF)^{-1}$ and so is nonsingular (i.e. it has non-zero determinant). Hence $d\phi(\tilde{\mathbf{x}}) = 0$ if, and only if, $d\phi(\mathbf{x}) = 0$. □

We assume throughout the point charges are located at distinct points collinear on the x-axis $\mathcal{L}$. Choosing a parametrization of $\mathcal{L}$ we label the point charges in the order we meet them as we vary the parameter. The first point charge, located at $\mathbf{x}_1$, is called the outermost point charge, and we change coordinates if necessary to ensure this happens at $(0, 0, 0)$ with all subsequent charges on the positive x-axis.

Proposition 2.3. *Suppose that either (i) all q_i have the same sign or (ii) q_1 has the opposite sign to $q_2, \ldots, q_n$. Then the critical points of ϕ also lie on $\mathcal{L}$.*

Proof. Choosing suitable coordinates (i.e. scaling and rotating $\mathcal{L}$ appropriately), the point charges may be assumed to lie at the points $(x_1, 0, 0) = (0, 0, 0)$ and $(x_i, 0, 0)$, $i = 2, \ldots, n$. Thus the potential function is given by

$$\phi(\mathbf{x}) = \sum_{i=1}^{n} \frac{q_i}{\sqrt{(x-x_i)^2+y^2+z^2}}, .$$

To find the gradient of $\phi(\mathbf{x})$, taking the derivatives with respect to x,y and z yields :

$$\phi_x = \sum_{i=1}^{n} \frac{-q_i(x-x_i)}{[(x-x_i)^2+y^2+z^2]^{\frac{3}{2}}}$$
$$\phi_y = y\sum_{i=1}^{n} \frac{-q_i}{[(x-x_i)^2+y^2+z^2]^{\frac{3}{2}}}$$
$$\phi_z = z\sum_{i=1}^{n} \frac{-q_i}{[(x-x_i)^2+y^2+z^2]^{\frac{3}{2}}}.$$

(i) Setting $\phi_y = \phi_z = 0$, it is apparent that $y = z = 0$ and the result is established since the sum on the right-hand-side consists of non-zero terms each with the same sign.
(ii) Setting $\phi_y = \phi_z = 0$, it is apparent that y = z = 0 and the result is established unless

$$\sum_{i=1}^{n} \frac{-q_i}{[(x-x_i)^2+y^2+z^2]^{\frac{3}{2}}} = 0 \implies \sum_{i=1}^{n} \frac{-q_i x}{[(x-x_i)^2+y^2+z^2]^{\frac{3}{2}}} = 0.$$

Feeding this into the equation $\phi_x = 0$ and noting that $x_1 = 0$ yields

$$\sum_{i=2}^{n} \frac{q_i x_i}{[(x-x_i)^2+y^2+z^2]^{\frac{3}{2}}} = 0$$

Since $x_i \in (0, \infty)$ and all the q_i with $i \geq 2$ have the same sign, this is a contradiction. Hence $y = z = 0$ and the result is established. □

In case (i), the following elementary argument establishes precisely the number of critical points and implies Maxwell's conjecture holds.

Proposition 2.4. *Suppose there are n collinear point charges with all charges having the same sign. Then ϕ has $n-1$ critical points.*

Proof. Assume, without loss of generality, all the q_i are positive. Applying Proposition 2.3, the critical points of ϕ lie on the x-axis $\mathcal{L}$. Since $y = z = 0$ we can rewrite $\phi(\mathbf{x})$ as a function of x along, yielding

$$\phi(x) = \sum_{i=1}^{n} \frac{q_i}{|x-x_i|}.$$

Assume $x \in (x_{j-1}, x_j)$, where $j = 2, \ldots, n$. Then

$$\phi(x) = \sum_{i=1}^{j-1} \frac{q_i}{x - x_i} + \sum_{i=j}^{n} \frac{q_i}{x_i - x}$$
$$\phi'(x) = \sum_{i=1}^{j-1} \frac{-q_i}{(x - x_i)^2} + \sum_{i=j}^{n} \frac{q_i}{(x_i - x)^2}$$
$$\phi''(x) = \sum_{i=1}^{j-1} \frac{2q_i}{(x - x_i)^3} - \sum_{i=j}^{n} \frac{2q_i}{(x_i - x)^3} > 0.$$

Hence $\phi''(x) > 0$ on (x_{j-1}, x_j) and thus has at most one critical point on the interval (x_{j-1}, x_j). However,

$$\lim_{x \to x_{j-1}^-} \phi'(x) = -\lim_{x \to x_j^+} \phi'(x) = \infty$$

so the intermediate value theorem implies there is at least one critical point on this interval, and hence there is exactly one critical point on each interval between the two critical points. This yields $n - 1$ critical points. It is easy to see there are no critical points if $x < x_1 = 0$ or $x > x_n$ as $\phi' \neq 0$ on these intervals, and the claim is established. □

Of course, since $n - 1 < (n - 1)^2$, this implies Maxwell's conjecture.

2.2. **Preliminaries for the Proof of Theorem B.** A key ingredient will be the following classical result concerning the real roots of polynomials. For the purposes of this paper, a polynomial will always be assumed to have real coefficients and be written with the powers in descending order, e.g. $x^6 - 5x^4 + 2x - 3$.

Lemma 2.5. *(Descartes' Rule of Signs) The number of positive real roots of a polynomial $p(x)$ is either the number of sign changes between consecutive (nonzero) coefficients, or is less than this by an even natural number. The number of negative real roots can be found by applying this rule to $p(-x)$.*

The proof is via induction. See [1] for further details. Important special cases are when the number of sign changes are either zero or one. If it is zero, we know the polynomial has no roots, and if it is one then it follows that there is exactly one root.

3. Proof of Theorem A

Proof of Theorem A. We follow the proof from Proposition 2.4. There are $n + 1$ intervals. On each interval we solve $\phi'(x) = 0$ by taking common denominators. This results in a polynomial of degree $2(n - 1)$ whose roots are precisely the critical points of ϕ. So on each interval there are at most $2(n - 1)$ roots and the result follows.

□

4. Proof of Theorem B

4.1. **Setup.** In this section we will restrict to the case where $n = 4$ and assume the outermost charge q_1 is located at the origin with $-q_1 = q_2 = q_3 = q_4$. Moreover without loss of generality we assume $-q_1 = 1$ and all the other charges lie on the positive x-axis. Applying Proposition 2.3 (ii) we know that $\phi_x = 0$ and $y = z = 0$. Following Proposition 2.4, we know $\mathbf{x} = (x, 0, 0)$ is a critical point of $\phi(x)$ with $x \in (x_{j-1}, x_j)$ if, and only if,

$$\phi'(x) = \sum_{i=1}^{j-1} \frac{-q_i}{(x - x_i)^2} + \sum_{i=j}^{4} \frac{q_i}{(x_i - x)^2} = 0.$$

Here $j = 2, \ldots, 4$. When $x \in (-\infty, 0)$ and $x \in (x_4, \infty)$ this formula is also correct if suitably interpreted. To simplify the algebra further, we will rescale if necessary so that $x_2 = 1$. This rescaling is a change of coordinates in $\mathbb{R}^3$ and so does not affect the computation of the number of critical points. To make the polynomials which follow easier to read, we will relabel $x_3 = \alpha$ and $x_4 = \beta$.

Now, we notice that the sign of each term on the right-hand-side will depend on the position of x. As we have seen in the proof of Theorem A, the sign distribution will generate a different polynomial of degree at most six, yielding a bound of at most 30 roots. We now will break the argument into cases, depending on the position of $\mathbf{x}$ with respect to the charges.

(1) Suppose $x < 0$. Then x is a critical point of ϕ if, and only if,

$$\frac{-1}{x^2} + \frac{1}{(x-1)^2} + \frac{1}{(x-\alpha)^2} + \frac{1}{(x-\beta)^2} = 0$$

with $x < 0$.

(2) Suppose $0 < x < 1$. Then x is a critical point of $\phi_x(x)$ if, and only if,

$$\frac{1}{x^2} + \frac{1}{(x-1)^2} + \frac{1}{(x-\alpha)^2} + \frac{1}{(x-\beta)^2} = 0.$$

Hence there are no critical points of ϕ is this interval and the analysis of this case is finished.

(3) Suppose $1 < x < \alpha$. Then x is a critical point of ϕ if, and only if,

$$\frac{1}{x^2} - \frac{1}{(x-1)^2} + \frac{1}{(x-\alpha)^2} + \frac{1}{(x-\beta)^2} = 0$$

with $\alpha < x < \beta$.

(4) Suppose $\alpha < x < \beta$. Then x is a critical point of ϕ if, and only if,

$$\frac{1}{x^2} - \frac{1}{(x-1)^2} - \frac{1}{(x-\alpha)^2} + \frac{1}{(x-\beta)^2} = 0$$

with $\alpha < x < \beta$.

(5) Suppose $x > \beta$. Then x is a critical point of ϕ if, and only if,

$$\frac{1}{x^2} - \frac{1}{(x-1)^2} - \frac{1}{(x-\alpha)^2} - \frac{1}{(x-\beta)^2} = 0$$

with $x > \beta$.

4.2. **Analysis of Cases (1), (3), (4), and (5).** It is necessary to consider combinations of the following polynomials:

$$\begin{aligned} f(x) &= (x-1)^2(x-\alpha)^2(x-\beta)^2 \\ g(x) &= x^2(x-\alpha)^2(x-\beta)^2 \\ r(x) &= x^2(x-1)^2(x-\alpha)^2 \\ s(x) &= x^2(x-1)^2(x-\beta)^2. \end{aligned}$$

Expanding these polynomials yields

$$\begin{aligned} f(x) &= x^6 - ux^5 + vx^4 - tx^3 + qx^2 - jx + \alpha^2\beta^2 \\ g(x) &= x^6 - (2\alpha+2\beta)x^5 + (\alpha^2+4\alpha\beta+\beta^2)x^4 - (2\alpha^2\beta+2\alpha\beta^2)x^3 + \alpha^2\beta^2x^2 \\ r(x) &= x^6 - (2\alpha+2)x^5 + (\alpha^2+4\alpha+1)x^4 - (2\alpha^2+2\alpha)x^3 + \alpha^2x^2 \\ s(x) &= x^6 - (2\beta+2)x^5 + (\beta^2+4\beta+1)x^4 - (2\beta^2+2\beta)x^3 + \beta^2x^2, \end{aligned}$$

where

$$\begin{aligned} u &= 2\alpha+2\beta+2 \\ v &= \beta^2+4\alpha\beta+4\beta+\alpha^2+4\alpha+1 \\ t &= 2\alpha\beta^2+2\beta^2+2\alpha^2\beta+8\alpha\beta+2\beta+2\alpha^2+2\alpha \\ q &= \alpha^2\beta^2+4\alpha\beta^2+\beta^2+4\alpha^2\beta+4\alpha\beta+\alpha^2 \\ j &= 2\alpha^2\beta^2+2\alpha\beta^2+2\alpha^2\beta. \end{aligned}$$

With this notation, the polynomials are as follows. For Case (1), the rational function has a zero precisely when $(-f+g+r+s)(x) = 0$ with x negative, which simplifies to

$$2x^6-(2\alpha+2\beta+2)x^5+(\alpha^2+\beta^2+1)x^4+8\alpha\beta x^3-(4\alpha\beta^2+4\alpha^2\beta+4\alpha\beta)x^2+jx-\alpha^2\beta^2 = 0.$$

For Case (3), the rational function has a zero precisely when $f-g+r+s = 0$, which simplifies to

$$\begin{aligned} 0 =& 2x^6 - (2\alpha+2\beta+6)x^5 + (\beta^2+8\beta+\alpha^2+8\alpha+3)x^4 \\ & - (4\beta^2+8\alpha\beta+4\beta+4\alpha+4\alpha^2)x^3 \\ & + (4\alpha\beta^2+4\alpha^2\beta+4\alpha\beta+2\alpha^2+2\beta^2)x^2 \\ & - jx + \alpha^2\beta^2 \end{aligned}$$

with $1 < x < \alpha$.

For Case (4), the rational function has a zero precisely when $f - g - r + s = 0$, which simplifies to

$$\begin{aligned}0 =& 2(\alpha - \beta - 1)x^5 + (\beta^2 - \alpha^2 + 8\beta + 1)x^4 - (4\beta^2 + 8\alpha\beta + 4\beta)x^3 \\ &+ (4\alpha\beta^2 + 2\beta^2 + 4\alpha^2\beta + 4\alpha\beta)x^2 - jx + \alpha^2\beta^2\end{aligned}$$

with $\alpha < x < \beta$.

Finally in Case (5) the polynomial has a zero precisely when $f - g - r - s = 0$, which is (up to a minus) the same case as Case 1. Hence we seek to solve

$$2x^6 - (2\alpha + 2\beta + 2)x^5 + (\alpha^2 + \beta^2 + 1)x^4 + 8\alpha\beta x^3 - (4\alpha\beta^2 + 4\alpha^2\beta + 4\alpha\beta)x^2 + jx - \alpha^2\beta^2 = 0$$

but now with $x > \beta$.

4.3. **Analysis of Case 1.** We claim there is one negative real root of the polynomial $z = -f + g + r + s$. This must have one negative root by the Intermediate Value Theorem, since $z(0) < 0$ and $z(x) \to \infty$ as $x \to -\infty$. In fact, if we apply Descartes rule of signs to estimate the number of negative zeros, we obtain that $z(-x)$ is

$$2x^6 + (2\alpha + 2\beta + 2)x^5 + (\alpha^2 + \beta^2 + 1)x^4 - 8\alpha\beta x^3 - (4\alpha\beta^2 - 4\alpha^2\beta + 4\alpha\beta)x^2 - jx - \alpha^2\beta^2.$$

This has one sign change so this proves there is exactly one negative root of z as required.

4.4. **Analysis of Case (3).** Applying Descartes rule of signs to the polynomial $z(x) = (f - g + r + s)(x)$ does not yield any information in this case: it tells us there are either six, four, two or zero positive roots of the polynomial (and no negative roots). So we conclude there are at most six critical points of ϕ with $x \in (1, \alpha)$.

4.5. **Analysis of Case (4).** Applying Descartes's rule of signs to

$$\begin{aligned}z(x) =& 2(\alpha - \beta - 1)x^5 + (\beta^2 - \alpha^2 + 8\beta + 1)x^4 - (4\beta^2 + 8\alpha\beta + 4\beta)x^3 + (4\alpha\beta^2 + 2\beta^2 \\ &+ 4\alpha^2\beta + 4\alpha\beta)x^2 - jx + \alpha^2\beta^2\end{aligned}$$

we obtain there are at either five, three or one positive real roots. Hence, in a worst-case scenario, there are five roots in $(1, \alpha)$.

4.6. **Analysis of Case (5).** Here the argument presented in Case (1) does not work as Descartes' rule of signs is not immediately applicable, as the reader can check directly. To remedy this, we apply the change of coordinates

$$x \to \tilde{x} = \beta - (\beta - \alpha)x, \quad \tilde{y} = y, \quad \tilde{z} = z.$$

The reader may easily check the determinant of the corresponding Jacobian is $\alpha - \beta \neq 0$, so it is invertible and hence this transformation is indeed a change of coordinates. From Proposition 2.2, being a critical point is independent of coordinate system. Computing in the new coordinate system, $\phi(\tilde{\mathbf{x}})$ is the same polynomial as Case (1), but in the $\tilde{x}$ variable (with α replaced with $\tilde{\alpha}$ etc.). To see this, note that $\beta < x < \iff \tilde{x} < 0$. In these coordinates, ϕ has a critical point if, and only if, the polynomial $z(\tilde{x}) = (f + g + r - s)(\tilde{x}) = 0$ with $\tilde{x} < 0$. Notice the minus sign now lies before the last term since the point charge with opposite sign now lies at $\tilde{\beta}$. Expanding this polynomial in the same fashion as above yields

$$\begin{aligned} z(\tilde{x}) = 2\tilde{x}^6 &- (6\tilde{\alpha} + 2\tilde{\beta} + 2)\tilde{x}^5 + (3\tilde{\alpha}^2 + \tilde{\beta}^2 + 8\tilde{\alpha}\tilde{\beta} + 8\tilde{\alpha} + 1)\tilde{x}^4 \\ &- (4\tilde{\alpha}^2\tilde{\beta} + 4\tilde{\alpha}\tilde{\beta}^2 + 4\tilde{\alpha}^2 + 8\tilde{\alpha}\tilde{\beta} + 4\tilde{\alpha})\tilde{x}^3 \\ &+ (2\tilde{\alpha}^2\tilde{\beta}^2 + 4\tilde{\alpha}\tilde{\beta}^2 + 4\tilde{\alpha}^2\tilde{\beta} + 4\tilde{\alpha}\tilde{\beta} + 2\tilde{\alpha}^2)\tilde{x}^2 \\ &- \tilde{j}\tilde{x} + \tilde{\alpha}^2\tilde{\beta}^2. \end{aligned}$$

Solving $z(\tilde{x}) = 0$ with $\tilde{x} < 0$ yields, upon applying Descartes' rule of signs in the same fashion, that there is no critical point in this interval.

4.7. **Conclusion of the Proof.**

Proof of Theorem B. This now follows immediately from combining established facts. There is one critical point in $(-\infty, 0)$, none in $(0, 1)$ and (β, ∞), at most five in (α, β), and at most six in $(1, \alpha)$. Hence there are at most twelve critical points. $\square$

5. Final Remarks

We conclude with some final remarks. The obvious way to improve the bound of 13 we obtain is to apply Sturm's lemma to precisely calculate the number of roots inside each interval, but the algebra becomes very difficult. This is a project for the future.

At first glance, it is somewhat curious that Descartes' rule of signs does not work in Case (5), but after a change of coordinates it can be made to work. This can be explained by observing that the change of coordinates presented changes the direction of the positive x-axis. To illustrate, consider the case where there are two charges in $(0, \beta)$, and three in (β, ∞). Then there are five positive roots in the original coordinate system on the positive x-axis, but only two in the $\tilde{x}$ coordinates. This example illustrates that it is possible to change the number of positive and negative roots by applying changes of variables.

It is also natural to ask how much further the collinear case can be investigated. We hope to take this project up into the future. In particular, it is clear that the assumption all the charges lie in a line allow us to reduce the equations to finding roots of polynomials of higher degree, and Descartes' Rule of Signs is a useful tool to help us compute the number of roots.

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