Quasi-Einstein Manifolds on Sphere Bundles

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Summary

We construct new solutions to the quasi–Einstein metrics. Via standard techniques, the main application of this result is that it yields new solutions to Einstein's equations. Such spaces were first studied by Einstein as models for the universe. Today they are still objects of central importance in both physics and mathematics, serving as fundamental building blocks in general relativity.

Warped Products

The Euclidean plane is a product: $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. We measure the squared length of a vector $\mathbf{v} = (x, y)$ by adding the squares of the lengths of the projections onto each factor:

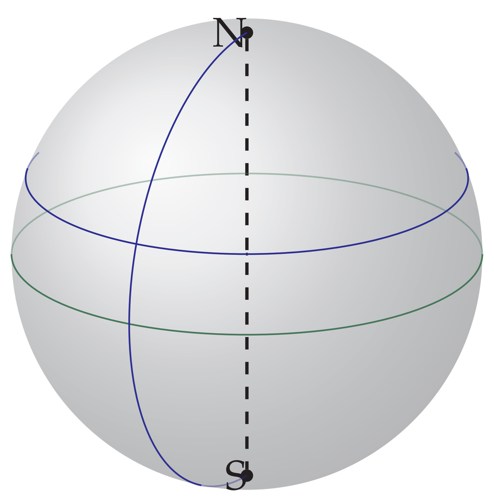
$$\|v\|^2 = x^2 + y^2.$$

More generally, if (M, g) and (H, h) are Riemannian manifolds the product manifold is $(M \times N, g + h)$.

Definition 0.1 *A* **warped product** *metric on $M \times N$ is given as $g + e^f h$, where $f \in C^\infty(M)$.*

Example 0.2 *Taking $f = 0$ yields usual products, so this is a generalization.*

Example 0.3 *Take $(0, \pi) \times \mathbb{S}^1$ with metric $dt^2 + \sin^2(t)d\theta^2$). We obtain a description of the round sphere $\mathbb{S}^2$ with the north and south poles removed.*



Applications

The fundamental reason we care about the quasi-Einstein equations is

Theorem 0.4 *Let $(\mathbb{S}^2, h)$ denote the round sphere. Then $(M \times \mathbb{S}^2, g + e^f h)$ is Einstein $\iff$ (M, g, f) is quasi-Einstein.*

In fact $\mathbb{S}^2$ can be any Einstein manifold. Our aim therefore is to find a solution to the quasi-Einstein equations.

Setup

Warped products are now used in a second way: to find an explicit solution to the quasi-Einstein equations .

Lemma 0.5 *Let (H, h) be a Riemannian manifold. For the warped product metric $g = dt^2 + f^2(t)h$ on $I \times H$ has (one of its) Ricci curvature terms given by $\frac{-f''}{f}$.*

Example 0.6 *For $(0, \pi) \times \mathbb{S}^1$, $dt^2 + \sin^2 t d\theta^2$, there is only one curvature terms:*

$$\frac{-f''}{f} = \frac{-(-\sin t)}{\sin t} = 1,$$

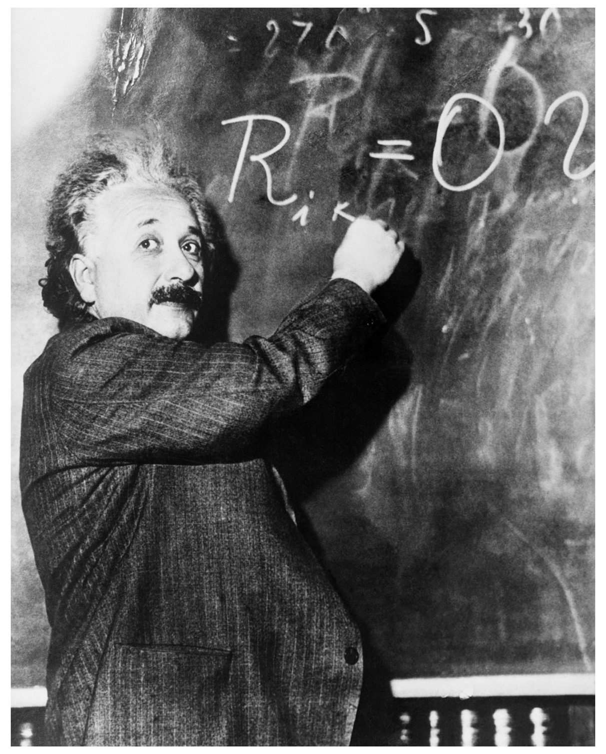
This is the "round" sphere, which is curved the same for all values of t.

Main Theorem

Theorem 0.7 *Let M be a product of Fano Kähler–Einstein manifolds (M_i, h_i) with Ric $= p_i h_i$. Then the $\mathbb{S}^2$-bundle over M admits quasi–Einstein metrics if $0 < |q_i| < p_i$, where the q_i are integers which determine the Euler class.*

Remarks on Main Theorem

The Euler class measure how twisted the bundles is: a strip of paper has zero Euler class, but a Möbius band has non-zero class.



The photograph shows Einstein writing out his equation: note the Ricci curvature term. We will rewrite our Ricci cuvature term as follows. Let $\alpha(s) = f^2(t)$ and $ds = f(t)dt$. Differentiating α,

$$\alpha'(s) = 2f(t)\dot{f}(t)\frac{dt}{ds} = 2f(t)\dot{f}(t)\frac{1}{f(t)} = 2\dot{f}(t) \implies$$

$$\alpha''(s) = 2\ddot{f}(t)\frac{dt}{ds} = 2\frac{\ddot{f}}{f} \implies \boxed{-\frac{\alpha''(s)}{2} = -\frac{\ddot{f}}{f}}.$$

This explains the first term we encounter in the quasi-Einstein equations below; it corresponds to one of the Ricci curvature.

Quasi-Einstein Equations

Let s be the coordinate on $I = (0, l)$ such that $ds = f(t)dt$, $\alpha(s) = f^2(t)$, $\beta_i(s) = g_i^2(t)$, $\phi(s) = v(t)$ and $V = \Pi_{i=1}^{i=r} g_i^{2n_i}(t)$. We choose our anti-derivative so that s ranges over the interval $[0, s_*]$. Under the coordinate changes, the quasi-Einstein equations are given by:

$$\frac{1}{2}\alpha'' + \frac{1}{2}\alpha'(\log V)' + \alpha\sum_{i=1}^{r} n_i\left(\frac{\beta_i''}{\beta_i} - \frac{1}{2}\left(\frac{\beta_i'}{\beta_i}\right)^2\right) + m\left(\frac{\alpha\phi''}{\phi} + \frac{\alpha'\phi'}{2\phi}\right) = \frac{\epsilon}{2}, \quad (1)$$

$$\frac{1}{2}\alpha'' + \frac{1}{2}\alpha'(\log V)' - \alpha\sum_{i=1}^{r}\frac{n_i q_i^2}{2\beta_i^2} + m\frac{\alpha'\phi'}{2\phi} = \frac{\epsilon}{2}, \quad (2)$$

$$\frac{1}{2}\frac{\alpha'\beta_i'}{\beta_i} + \frac{1}{2}\alpha\left(\frac{\beta_i''}{\beta_i} - \left(\frac{\beta_i'}{\beta_i}\right)^2\right) + \frac{1}{2}\frac{\alpha\beta_i'}{\beta_i}(\log V)' - \frac{p_i}{\beta_i} + \frac{q_i^2\alpha}{2\beta_i^2} + m\frac{\alpha}{2}\frac{\beta_i'\phi'}{\beta_i\phi} = \frac{\epsilon}{2} \quad (3)$$

$$\phi\left(\phi''\alpha + \frac{\phi'\alpha'}{2}\right) + \phi\phi'\left(\frac{\alpha'}{2} + (\log V)'\alpha\right) + (m-1)(\phi')^2\alpha - \frac{\epsilon}{2}\phi^2 = \mu. \quad (4)$$

The solutions to the system of ordinary differential equations are

$$\phi(s) = \kappa_0(s + \kappa_1), \quad \beta_i(s) = A_i(s + \kappa_0)^2 - \frac{q_i^2}{4A_i}, \quad \alpha(s) = V^{-1}(s + \kappa_0)^{1-m}\int_0^s V(t + \kappa_0)^{m-2}\left(E + \frac{\epsilon}{2}(t + \kappa_0)^2\right)dt.$$

Remark: One can solve for $\alpha(s)$ using the tools from Math 250B since the system boils down to solving a first order linear differential equation,

$$\alpha' + \alpha\left((\log V)' + m(\log\phi)' - \frac{1}{s + s_0}\right) = \frac{\epsilon}{2}(s + \kappa_0) + \frac{E}{(s + k_0)}$$

where E is constant. Applying the integrating factor, we get

$$I = e^{\int(\log V)' + \frac{(m-1)}{(s+\kappa_0)}ds} = e^{\log V + (m-1)\log(s+\kappa_0)} = V(s + \kappa_0)^{m-1}.$$

Proof

Hall's proof has an issue when he determnines κ_0 and s_* as functions of E. The proof presented does not work for the case of $\mathbb{S}^2$-bundles but only in the case when blowdowns are allowed. We find a different way of determining these numbers as functions of E.

Lemma 0.8 *With the same notation as above;*

1. *If $\alpha(0) = 0$, then κ_0 is a root of the quadratic $\frac{1}{2}x^2 + 2x - E$.*
2. *If $\alpha(s_*) = 0$, then $-\kappa_0 - s_*$ is a root of the quadratic $\frac{1}{2}x^2 + 2x - E$.*

This can be proved by plugging in the assumptions into equation (3).

Extending to a Smooth Metric

Given the rationally symmetric metric on $(0, \pi) \times \mathbb{S}^{n-1}(1)$,

$$g = dt^2 + f^2(t)ds_{n-1}^2,$$

we wish it to extend smoothly at $t = 0$. Using the coordinate change $x = ts$ with $x \in \mathbb{R}^n$, $t > 0$, and $s \in \mathbb{S}^{n-1}(1)$, we require that the functions

$$\frac{f^2(t)}{t^2}, \ \frac{1}{t^2}\left(1 - \frac{f^2(t)}{t^2}\right)$$

to be smooth at $t = 0$. To do so, we just need the functions to be continuous at $t = 0$ or else it will not be differentiable. In other words,

$$\lim_{t\to 0}\frac{f^2(t)}{t^2}, \ \lim_{t\to 0}\frac{1}{t^2}\left(1 - \frac{f^2(t)}{t^2}\right) \in \mathbb{R}.$$

For the first limit, since the denominator goes to 0 as t approaches zero, we need $f(0) = 0$. Similarly, for the second limit, we need to assume that $\dot{f}(0) = 1$. Like the first limit, $\lim_{t\to 0} t^2 = 0$ so we need $\lim_{t\to 0}(1 - f^2(t)/t^2) = 1$. Notice that

$$\lim_{t\to 0}\frac{f^2(t)}{t^2} = \left(\lim_{t\to 0}\frac{f(t)}{t}\right)^2 = \left(\lim_{t\to 0}\frac{f(t) - f(0)}{t - 0}\right)^2 = (\dot{f}(0))^2 = 1$$

which is desired. With our choice of $\alpha(s) = f^2(t)$, we get that $\alpha'(0) = 2$.

Future Research

Into the future, we aim to study the quasi-Einstein equations in the noncompact case and find new solutions there. This is a natural next question.

References

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