Circumscribing Archimedean Solids With A Regular Tetrahedron

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Introduction

In studying any family of mathematical objects, a fundamental issue is to understand how one object can “sit inside” another object in the family, preserving the mathematical structure. We are concerned with convex uniform polyhedrons. Two famous families of polyhedra live in this class: the Platonic and Archimedean solids, as well as the prisms and antiprisms. Our main result geometrically characterizes the famed Archimedean solids among the convex uniform polyhedra by studying how they sit inside a regular tetrahedron.

Main Theorem: Characterization of Archimedean solids

A convex uniform polyhedron P can be circumscribed by a regular Tetrahedron so that four faces of P lie on the four faces of the Tetrahedron if, and only if, P is either (i) an Icosahedron, (ii) an Octahedron, or (iii) an Archimedean solid.

Platonic and Archimedean Solids

The famous Platonic solids (the Tetrahedron, Cube, Octahedron, Dodecahedron, and Icosahedron) are the only 5 convex regular polyhedra. These are the most symmetric and most beautiful of the polyhedra. Regularity means that each face is a congruent regular polygon, and the same number of faces meet at each vertex. Relaxing the requirement that the faces are congruent expands this list to the class of uniform polyhedra.

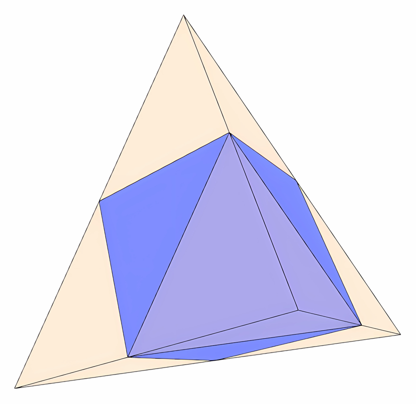
The Archimedean solids are a subset of the convex uniform polyhedra. The precise definition of the Archimedean solids has been the subject of much literature, and there are various subtle issues: see [Grünbaum]. In [Pugh], our main theorem is taken as the defining property of Archimedean solids: Pugh states that Archimedean solids should be defined as convex uniform polyhedra which are not regular and can be circumscribed by a regular tetrahedron so that four of its faces lie on the tetrahedron. However, no proof or details are given in anywhere in the literature. The point of our main Theorem is to provide a proof which shows that Pugh's definition/characterization is well-defined.

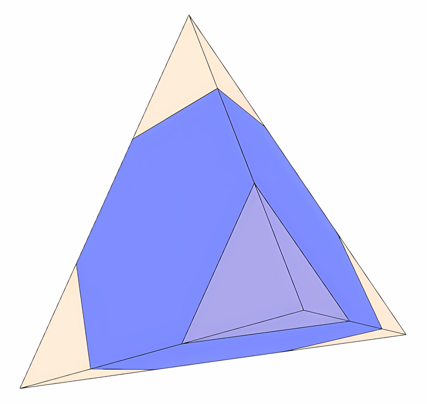
Brute Force Proof

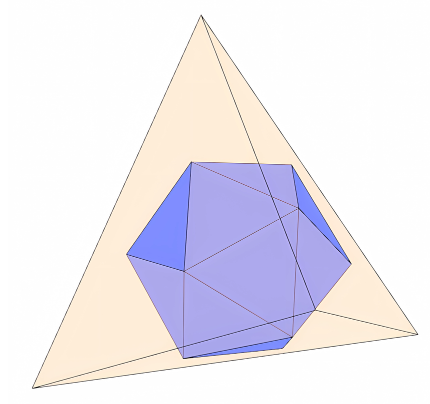
In order to verify any legitimacy of this claim, we first made use of a brute force computation. The Cartesian coordinates of each vertex are known (up to scale). We need to check that each solid has a set of four faces such that each pair of these faces have the appropriate dihedral angle of a tetrahedron.

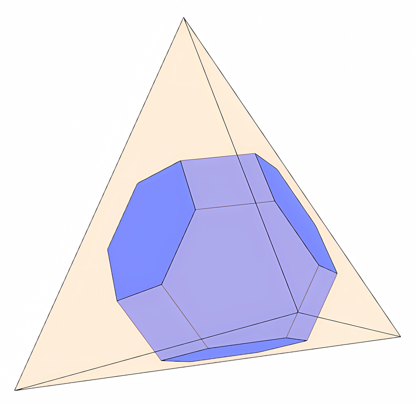
```
for file in os.listdir('Solids'):
    with open('Solids/'+file, 'r') as f:
        data = json.load(f)
        midpoints = list()
        for face in data['faces']:
            if len(face) % 3 == 0:  #only consider faces with 3-fold symmetry
                midpoint = [0,0,0]
                for vertex in face:
                    point = data['vertices'][vertex]
                    midpoint[0] += point[0]
                    midpoint[1] += point[1]
                    midpoint[2] += point[2]
                midpoint = [p/len(face) for p in midpoint]
                midpoints.append(midpoint)
        possible_sets = combinations(midpoints, 4)
        for comb in possible_sets:
            angles = list()
            for pair in combinations(comb, 2):
                angles.append(angle_between(pair[0], pair[1]))
            if all(x == 1.91063 for x in angles):  # angle for tetrahedron
                print('It works!', file)
```

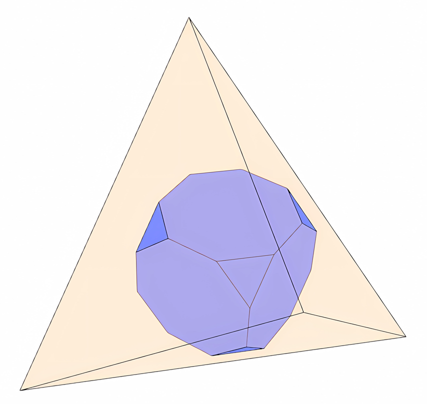
All possible Embeddings

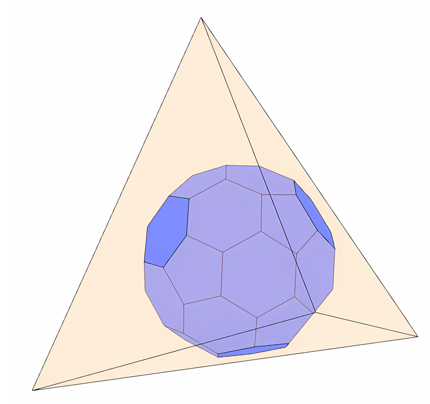

Octahedron

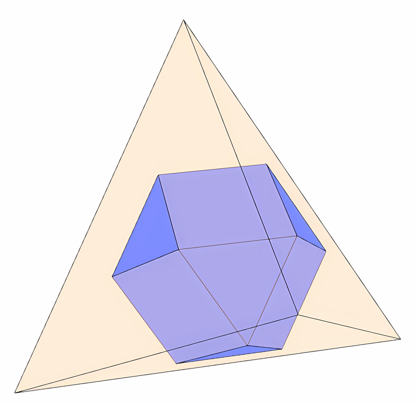

Truncated Tetrahedron

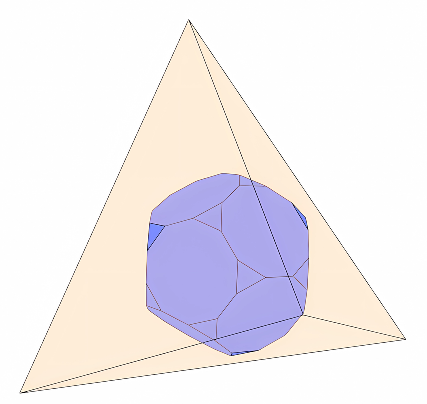

Icosahedron

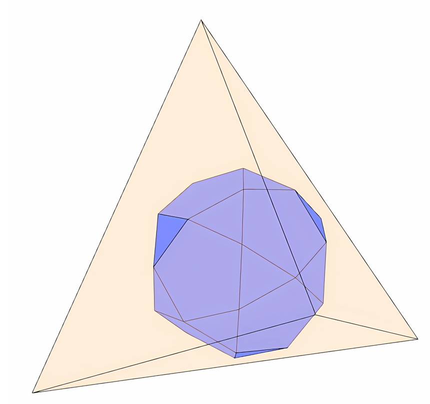

Truncated Octahedron

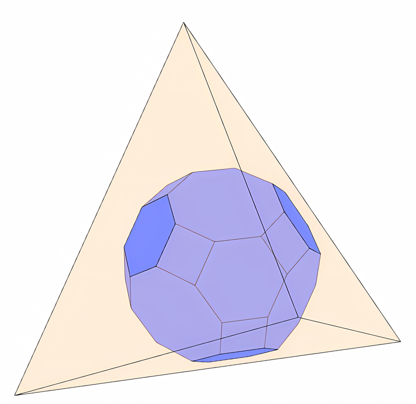

Truncated Cube

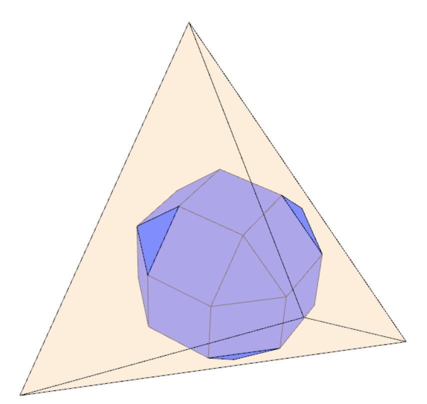

Truncated Icosahedron

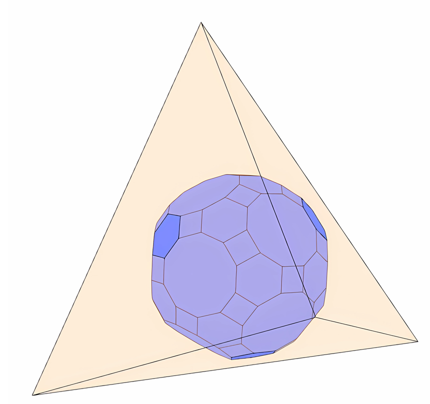

Cuboctahedron

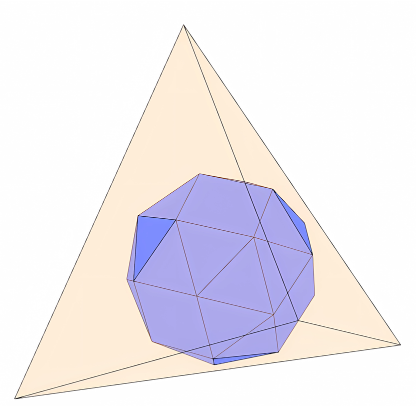

Truncated Dodecahedron

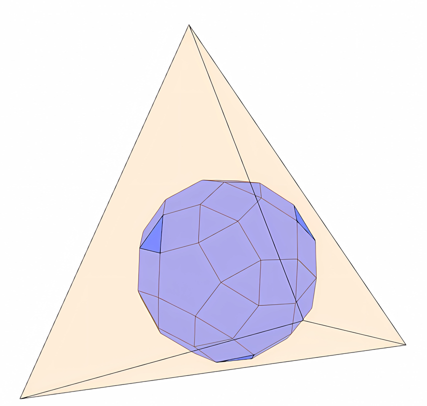

Icosidodecahedron

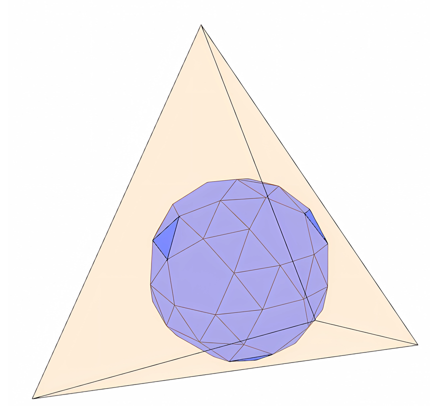

Truncated Cuboctahedron


Rhombicuboctahedron


Truncated Icosidodecahedron


Snub Cube

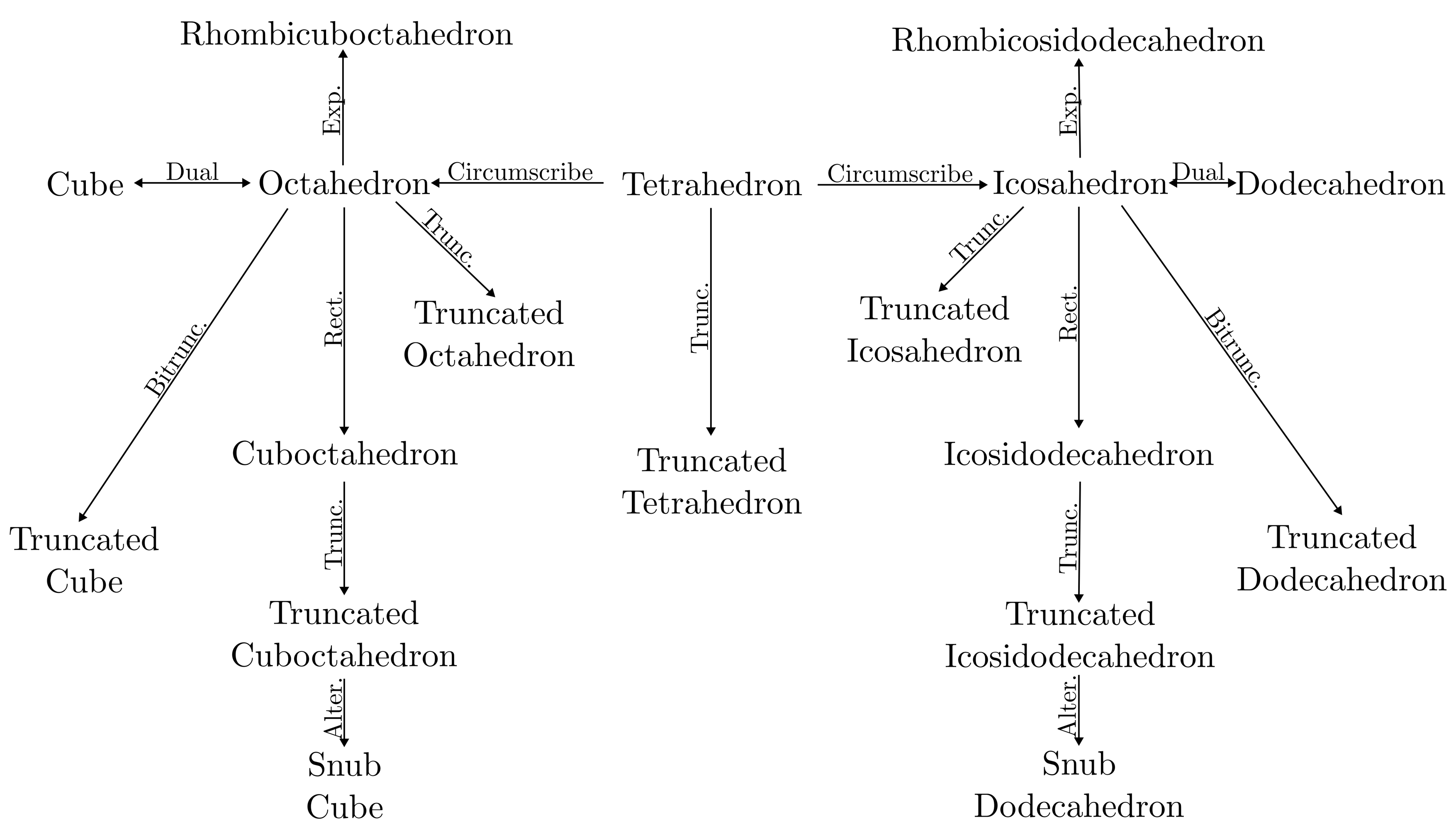

Rhombicosidodecahedron


Snub Dodecahedron

Conceptual Proof

A conceptual proof of this result can be given, which is much more satisfying mathematically. The main task is to show how each Archimedean solid can be circumscribed by a regular tetrahedron in the desired way. To prove this, we start by observing that every Archimedean solid can be constructed from a Platonic solid via a standard sequence of operations: one can *truncate* **T** (including bitruncation and rectification), *expand* **E**, or *snub* **S** (also called alternatation). The relationships are shown in the chart below. Starting at the tetrahedron, one can apply some combination of **T**,**E**, and **S** to arrive at any Archimedean solid. It is classically known that the Tetrahedron circumscribes the Octahedron and Icosahedron with our desired property (four faces of the Platonic solid lie on the tetrahedron). These are displayed in the first row of the images. The key step in our proof is to show that each operation **T**,**E**, and **S** does not affect these four faces. Hence any Archimedean solid which is built from the Octahedron or Icosahedron will also have this property. This covers all Archimedean solids.

Relationship



Truncation T: A sample of the proof

We explain one of the operations: truncation is the process of cutting a polyhedron along several planes. The plane by which we truncate is defined by moving along the edges a certain distance from the vertex. This distance is considered as a ratio of the edge.

Uniform truncation is truncation such that the resulting faces are all regular polygons. This forms five of the Archimedean solids: the truncated Tetrahedron, truncated Cube, truncated Octahedron, truncated Dodecahedron, and truncated Icosahedron.

The important feature of truncation is that it does not alter the angles of the original faces. Therefore, since the Octahedron and Icosahedron can each be circumscribed by a tetrahedron in the desired manner, so too can the truncated Octahedron and Icosahedron.

References

[Pugh] A. Pugh, *Polyhedra: a visual approach*, University of California Press, Berkeley, Calif.-London, 1976.

[Gru] B. Grünbaum, *An Enduring Error*, Elem. Math. 64 (2009), no. 3, 89–101.