A SIMPLE FORMULA FOR THE PICARD NUMBER OF K3 SURFACES OF BHK TYPE

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ABSTRACT. The BHK mirror symmetry construction stems from work Berglund and Hübsch [BH2], and applies to certain types of Calabi-Yau varieties that are birational to finite quotients of Fermat varieties. Their definition involves a matrix A and a certain finite abelian group G, and we denote the corresponding Calabi-Yau variety by $Z_{A,G}$. The transpose matrix A^T and the so-called dual group G^T give rise to the BHK mirror variety Z_{A^T,G^T}. In the case of dimension 2, the surface $Z_{A,G}$ is a *K3 surface of BHK type*.

Let $Z_{A,G}$ be a K3 surface of BHK type, with BHK mirror Z_{A^T,G^T}. Using work of Shioda [Shi2], Kelly shows in [Kel2] that the geometric Picard number $\rho(Z_{A,G})$ of $Z_{A,G}$ may be expressed in terms of a certain subset of the dual group G^T. We simplify this formula significantly to show that $\rho(Z_{A,G})$ depends only upon the degree of the mirror polynomial F_{A^T}.

1. INTRODUCTION

Let k be an algebraically closed field of characteristic $p \geq 0$, and let X be a smooth projective surface defined over k. The Picard number of X is defined as the rank of the Neron-Severi group of X:

$$\rho = \rho(X) = \operatorname{rank}_{\mathbb{Z}} \operatorname{NS}(X),$$

where $\operatorname{NS}(X)$ denotes the group of divisors on X modulo algebraic equivalence. The Picard number is a fundamental invariant of a surface, but in practice the determination of $\rho(X)$ can be difficult for a given surface X.

The case of K3 surfaces illustrates this point well, where one has $1 \leq \rho \leq 20$ in characteristic 0 and $1 \leq \rho \leq 22$ in positive characteristic. For instance, when $\rho(X) \geq 5$, one knows that X has an elliptic fibration, and when $\rho(X) \geq 12$, then such a fibration exists having a section (see for instance [Huy, §11.1]). If $\rho(X) \geq 17$, one may often deduce the existence of a nontrivial algebraic correspondence between X and an abelian surface [Mor]. K3 surfaces satisfying $\rho = 22$ (in positive characteristic) fall into the highly interesting class of *supersingular* K3 surfaces. This small selection of facts shows the importance of determining the Picard number of a K3 surface, but this task is usually far from straightforward. This may be inferred, for instance, from van Luijk's construction [vL] of the first explicit example of a K3 surface over $\mathbb{Q}$ with $\rho = 1$, a relatively recent feat within the much longer history of K3 surfaces.

In [Shi2], Shioda identifies a special class of surfaces for which the Picard number may be determined in a combinatorial manner. Any one of these surfaces X is birational to the quotient of a Fermat surface $X_0^d + X_1^d + X_2^d + X_3^d = 0$ by a finite group Γ, and this allows Shioda to obtain a formula of the form

$$\rho(X) = b_2(X) - \#(\mathfrak{I}_d(p) \cap H),$$

where $b_2(X)$ is the second Betti number of X, H is a certain subgroup of $(\mathbb{Z}/d)^4$ (determined by Γ), and $\mathfrak{I}_d(p)$ is a certain subset of $(\mathbb{Z}/d)^4$ (see §5). In this sense, the formula for $\rho(X)$ in [Shi2] is elementary and computable; however, the nontrivial amount of calculation required generally makes it difficult to predict the value of $\rho(X)$ without completing the full computation.

One interesting class of surfaces to which Shioda's formulas apply are *K3 surfaces of BHK type*. These are the 2-dimensional examples of special types of Calabi-Yau varieties for which a notion of mirror symmetry was initiated by Berglund–Hübsch [BH2], and extended by Berglund–Henningson [BH1] and Krawitz [Kra]. Roughly speaking, a K3 surface of BHK type is determined by what we shall call an *adequate BHK pair* (A, G), where $A = (A_{ij})$ is a 4-by-4 matrix of nonnegative integers and G is a subgroup of $(\mathbb{Z}/d)^4$ (for some $d \geq 1$ determined by A); see §2 and §3 for the precise requirements on A and G. The matrix A yields a polynomial of the form

$$(1.1) \qquad F_A = \sum_{i=0}^{3} x_0^{A_{i0}} x_1^{A_{i1}} x_2^{A_{i2}} x_3^{A_{i3}},$$

one which turns out to be quasihomogeneous of some degree h and weights $\mathbf{q} = (q_0, q_1, q_2, q_3)$. This will define a hypersurface X_A of degree h in the weighted projective space $\mathbb{P}^3(\mathbf{q})$, and its minimal resolution $\tilde{X}_A$ will be a K3 surface. The group G yields a group of symplectic automorphisms on $\tilde{X}_A$, and hence the minimal resolution of the quotient $\tilde{X}_A/G$ is again a K3 surface $Z_{A,G}$; the latter surface is called a K3 surface of BHK type.

The BHK mirror construction starts by forming the transpose A^T, which in turn yields a second polynomial F_{A^T}. From G, one may also form the so-called *dual group* $G^T \subseteq (\mathbb{Z}/d)^4$ (see §4). When (A^T, G^T) is another adequate BHK pair, it gives rise to the K3 surface Z_{A^T, G^T}, which is the *BHK mirror surface* of $Z_{A,G}$.

In [Kel2], Kelly applies Shioda's formula to the BHK construction to reveal a striking relationship between the Picard numbers of the K3 surfaces $Z_{A,G}$ and Z_{A^T, G^T}, one that adds weight to the "mirror" adjective:

Theorem 1.1 (Kelly). *Let $Z_{A,G}$ and Z_{A^T, G^T} be K3 surfaces of BHK type that are BHK mirrors. Their Picard numbers are given by*

$$
\begin{aligned}
\rho(Z_{A,G}) &= 22 - \#(\mathfrak{I}_d(p) \cap G^T) \\
\rho(Z_{A^T, G^T}) &= 22 - \#(\mathfrak{I}_d(p) \cap G).
\end{aligned}
$$

Yet despite the appealing nature of the relation in Theorem 1.1, the precise values of $\rho(Z_{A,G})$ and $\rho(Z_{A^T, G^T})$ can still be difficult to anticipate without a fair amount of calculation. One arithmetically interesting demonstration of this is given by the following question: Noting that F_A has integral coefficients, if we fix the pair

(A, G) and allow the prime p to vary, how does $\rho(Z_{A,G})$ vary? As an intriguing example at the end of [Kel2] suggests, the answer is not obvious, but nevertheless appears to be simpler than one might expect from Shioda's formula.

Our main result simplifies the formulas in Theorem 1.1 and also answers the question of how the Picard number of $Z_{A,G}$ varies with p:

Theorem 1.2. *Let $Z_{A,G}$ and Z_{A^T, G^T} be K3 surfaces of BHK type that are BHK mirrors. Let h (resp. h_T) denote the degree of the quasihomogeneous polynomial F_A (resp. F_{A^T}). Over the algebraically closed field k, the Picard numbers of these surfaces are given as follows:*

(1) *If* $\operatorname{char} k = 0$ *then*

$$\rho(Z_{A,G}) = 22 - \phi(h_T)$$
$$\rho(Z_{A^T, G^T}) = 22 - \phi(h),$$

where ϕ denotes Euler's totient function.

(2) *If* $\operatorname{char} k = p > 0$ *with* $p \nmid d$ *then*

$$\rho(Z_{A,G}) = \begin{cases} 22 & \text{if } p^\ell \equiv -1 \pmod{h_T} \text{ for some } \ell \\ 22 - \phi(h_T) & \text{if } p^\ell \not\equiv -1 \pmod{h_T} \text{ for all } \ell \end{cases}$$

$$\rho(Z_{A^T, G^T}) = \begin{cases} 22 & \text{if } p^\ell \equiv -1 \pmod{h} \text{ for some } \ell \\ 22 - \phi(h) & \text{if } p^\ell \not\equiv -1 \pmod{h} \text{ for all } \ell \end{cases}$$

In particular, we have the following result:

Corollary 1.3. *The Picard number of $Z_{A,G}$ is independent of G.*

We note that, in characteristic 0, this corollary is immediate from a result of Inose [Ino, Cor. 1.2], which says that if one has complex K3 surfaces X, Y and a rational map $\pi \colon X \to Y$, then $\rho(X) = \rho(Y)$. Our proof will be independent of this fact, but see Remark 8.3 for more discussion of this result of Inose.

Another consequence is that the possible values of Picard numbers of K3 surfaces of BHK type are constrained only by the values of the ϕ-function:

Corollary 1.4. *In characteristic 0, the (geometric) Picard number of a K3 surface of BHK type may equal any even integer between 2 and 20, with the exception of 8.*

Proof. Given that the Picard number of a K3 surface in characteristic 0 lies between 1 and 20, part (1) of Theorem 1.2 says it suffices to find BHK mirror surfaces $Z_{A,G}$ and Z_{A^T, G^T} such that $\phi(h_T) \in \{2, 4, 6, \ldots, 20\} \setminus \{14\}$. In Table 3 of §9, we have listed one explicit example for each case. On the other hand, one may show that $\phi(k) \neq 14$ for every positive integer k, so the Picard number of a K3 surface of BHK type can never equal 8. $\square$

We note that, by part (2) of Theorem 1.2, the possibilities for the Picard number in characteristic $p > 0$ are the same as those listed in the Corollary, along with the additional possibility of having Picard number 22.

The proof of Theorem 1.2 proceeds by an analysis of the set $\mathfrak{I}_d(p) \cap G \subseteq (\mathbb{Z}/d)^4$, for a given BHK pair (A, G). For a certain subset $\mathfrak{A}_d \subseteq (\mathbb{Z}/d)^4$ (see §5), the

multiplicative group $(\mathbb{Z}/d)^{\times}$ acts upon $\mathfrak{A}_d \cap G$ by coordinatewise multiplication, and one may recast $\mathfrak{I}_d(p) \cap G \subseteq (\mathbb{Z}/d)^4$ as the union of those orbits of this action which possess a certain property. (This property of the orbits is described in terms of certain special elements in $\mathfrak{A}_d$; see §7. We note that this idea is also implicit in [Shi2, Cor. 2].) On the other hand, $\mathfrak{I}_d(p) \cap G$ also has a Hodge theoretic meaning, and in the context of K3 surfaces this allows one to deduce that there is at most one orbit having the aforementioned property (see Proposition 6.2). An explicit description of this one orbit then gives the theorem.

In the final section of the paper, we revisit the example from the end of [Kel2], compute the Picard numbers of K3 surfaces of BHK type arising from quartic surfaces and double sextics, and give explicit examples that verify Corollary 1.4.

Acknowledgements

2. BHK pairs

In this section we fix p to be either 0 or a (positive) prime integer.

The concepts discussed in this section might best be described as algebraic or combinatorial, the culmination being the notion of *BHK pair* in Definition 2.5. But our primary motivation for these concepts will be their application in the geometric setting discussed in the next section. In particular, the notion of an *adequate BHK pair* in Definition 3.1 will be a further refinement that incorporates additional desirable properties (such as mildness of certain singularities).

Definition 2.1. *Let $A = (A_{ij})_{0 \leq i,j \leq 3} \in M_{4 \times 4}(\mathbb{Z})$. We will say A is a* weighted Delsarte matrix *if it satisfies all of the following:*

  (1) *All entries of A are nonnegative*
  (2) *Each column has at least one zero*
  (3) *We have $p \nmid \det(A)$ (and in particular $\det(A) \neq 0$)*
  (4) *The vector $A^{-1}(1,1,1,1)^T$ consists of positive entries*

Now let k denote an algebraically closed field of characteristic p. Associated to A, we define the polynomial

$$F_A = \sum_{i=0}^{3} \prod_{j=0}^{3} x_j^{A_{ij}} = \sum_{i=0}^{3} x_0^{A_{i0}} x_1^{A_{i1}} x_2^{A_{i2}} x_3^{A_{i3}}.$$

Notice that if $\lambda \in k^\times$ and $\mathbf{n} = (n_0, n_1, n_2, n_3) \in \mathbb{Z}^4$, then

$$(2.1) \qquad F_A(\lambda^{n_0} x_0, \lambda^{n_1} x_1, \lambda^{n_2} x_2, \lambda^{n_3} x_3) \;=\; \sum_{i=0}^{3} \prod_{j=0}^{3} (\lambda^{n_j} x_j)^{A_{ij}}$$

$$(2.2) \qquad\qquad\qquad\qquad\qquad =\; \sum_{i=0}^{3} \left(\lambda^{\sum_{j=0}^{3} A_{ij} n_j} \prod_{j=0}^{3} x_j^{A_{ij}} \right)$$

$$(2.3) \qquad\qquad\qquad\qquad\qquad =\; \sum_{i=0}^{3} \left(\lambda^{(A\mathbf{n}^T)_i} \prod_{j=0}^{3} x_j^{A_{ij}} \right).$$

As a first consequence of this, let the vector in (4) be given by

$$(2.4) \qquad\qquad A^{-1}(1,1,1,1)^T = \left(\frac{q_0}{h}, \frac{q_1}{h}, \frac{q_2}{h}, \frac{q_3}{h} \right),$$

where each q_i and h are (positive) integers and $\gcd(q_0, q_1, q_2, q_3) = 1$. Putting $\mathbf{q} = (q_0, q_1, q_2, q_3)$, we have $A\mathbf{q}^T = (h, h, h, h)^T$ and thus by (2.3) the polynomial F_A satisfies

$$(2.5) \qquad\qquad F_A(\lambda^{q_0} x_0, \lambda^{q_1} x_1, \lambda^{q_2} x_2, \lambda^{q_3} x_3) = \lambda^h F_A(x_0, x_1, x_2, x_3)$$

for all $\lambda \in k^\times$. This says that F_A is *quasihomogeneous* of degree h and weight system $\mathbf{q} = (q_0, q_1, q_2, q_3)$.

Definition 2.2. *We will say that the weighted Delsarte matrix A satisfies the Calabi-Yau requirement if (with notation as above) we have $h = \sum_{i=0}^{3} q_i$.*

Remark 2.3. We note by (2.4) that A satisfies the Calabi-Yau requirement if and only if the entries of A^{-1} sum to 1.

For the rest of this section, we assume that we are working with weighted a Delsarte matrix A that also satisfies the Calabi-Yau requirement.

Next consider the following subgroup of $(k^\times)^4$:

$$\mathrm{Aut}(F_A) = \left\{ (\lambda_0, \lambda_1, \lambda_2, \lambda_3) \in (k^\times)^4 \mid F(\lambda_0 x_0, \lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3) = F_A(x_0, x_1, x_2, x_3) \right\}.$$

One may use (3) to show that $\mathrm{Aut}(F_A)$ is a subgroup of μ_d^4, where d is the smallest positive integer such that the matrix

$$(2.6) \qquad\qquad\qquad\qquad\qquad B = dA^{-1}$$

has integer entries and μ_d denotes the group of dth roots of unity in k. Moreover, one may show that $\# \mathrm{Aut}(F_A) = |\det(A)|$.

Lemma 2.4. *We have $h \mid d$, $d \mid \det(A)$, and $\det(A) \mid d^4$. In particular, the primes dividing d and $\det(A)$ are the same.*

Proof. One may characterize h as the positive generator of the following ideal in $\mathbb{Z}$:

$$\left\{ m \in \mathbb{Z} \mid mA^{-1}(1,1,1,1)^T \in \mathbb{Z}^4 \right\}.$$

By its definition in (2.6), d belongs to this ideal and hence we conclude $h \mid d$. Likewise, we may characterize d as the positive generator of the ideal

$$\left\{ m \in \mathbb{Z} \mid mA^{-1} \in M_{4\times4}(\mathbb{Z}) \right\},$$

and as $\det(A)A^{-1}$ is the adjugate matrix of A, we must have $d \mid \det(A)$. Finally, as $\mathrm{Aut}(F_A)$ has order $|\det(A)|$, it must divide the order of the group μ_d^4. The concluding remark in the statement of the lemma follows from the fact that $d \mid \det(A)$ and $\det(A) \mid d^4$. $\qquad\square$

Fix a primitive dth root of unity $\zeta \in k$. For any $\mathbf{a} = (a_0, a_1, a_2, a_3) \in (\mathbb{Z}/d)^4$, we find by (2.3) that

$$(\zeta^{a_0}, \zeta^{a_1}, \zeta^{a_2}, \zeta^{a_3}) \in \mathrm{Aut}(F_A) \quad \Longleftrightarrow \quad A\mathbf{a}^T = 0^T \text{ in } \mathbb{Z}/d.$$

Due to this, we will typically (but not always) use this additive notation:

$$(\zeta^{a_0}, \zeta^{a_1}, \zeta^{a_2}, \zeta^{a_3}) \in \mu_d^4 \quad \longleftrightarrow \quad \mathbf{a} = (a_0, a_1, a_2, a_3) \in (\mathbb{Z}/d)^4.$$

To reduce ambiguity between these notations, we will also use Greek variables for the multiplicative notation and Roman variables for the additive notation.

We now identify two subgroups of $\mathrm{Aut}(F_A)$. First (in our additive notation) let

$$\mathbf{j}_A = \frac{d}{h}\mathbf{q} = \left(\frac{q_0 d}{h}, \frac{q_1 d}{h}, \frac{q_2 d}{h}, \frac{q_3 d}{h} \right).$$

By (2.5), we see $\mathbf{j}_A \in \mathrm{Aut}(F_A)$, and we denote by $J_{F_A} = \langle \mathbf{j}_A \rangle$ the cyclic subgroup that it generates. Second, define

$$\mathrm{SL}(F_A) = \left\{ \mathbf{a} \in \mathrm{Aut}(F_A) \;\Big|\; \sum_{i=0}^{3} a_i = 0 \right\}.$$

(In the multiplicative notation, an element $(\lambda_0, \lambda_1, \lambda_2, \lambda_3) \in \mathrm{Aut}(F_A)$ belongs to $\mathrm{SL}(F_A)$ precisely when $\prod_{i=0}^{3} \lambda_i = 1$; by viewing $(\lambda_0, \lambda_1, \lambda_2, \lambda_3)$ as a diagonal matrix, one better comprehends this widely-used notation for this subgroup of $\mathrm{Aut}(F_A)$.) The Calabi-Yau condition on A implies that $J_{F_A} \subseteq \mathrm{SL}(F_A)$.

Definition 2.5. *A* BHK pair *will be a pair (A, G) such that:*

(1) *A is a weighted Delsarte matrix satisfying the Calabi-Yau condition.*
(2) *G is a subgroup satisfying $J_{F_A} \subseteq G \subseteq \mathrm{SL}(F_A)$.*

3. BHK surfaces of K3 type

From these algebraic ideas, we now pass to a more geometric setting, by using a BHK pair to form a hypersurface in weighted projective space. A standard reference for weighted projective spaces is [Dol].

Recall that k is a field of characteristic $p \geq 0$. We let (A, G) denote a BHK pair and assume the following further conditions:

- The weight system $\mathbf{q} = (q_0, q_1, q_2, q_3)$ satisfies $\gcd(q_i, q_j, q_\ell) = 1$ whenever $0 \leq i < j < \ell \leq 3$.
- $p \nmid q_i$ for all i.
- $p \nmid d$.

The polynomial F_A defines a hypersurface X_A (called a *weighted Delsarte surface*) of degree h in the weighted projective space $\mathbb{P}^3(\mathbf{q}) = \mathbb{P}^3(q_0, q_1, q_2, q_3)$, and X_A has at most cyclic quotient singularities. We assume that X_A is:

- *quasi-smooth*, meaning that the affine cone over X_A has the origin as its only singular point, and
- *well-formed*, meaning that the intersection of the (one-dimensional) singular locus of $\mathbb{P}^3(\mathbf{q})$ with X_A is zero-dimensional or empty.

Due to this, and the fact that A satisfies the Calabi-Yau requirement, the minimal resolution $\tilde{X}_A$ will be a K3 surface (see [Got]).

The group $\mathrm{Aut}(F_A)$ acts upon points of X_A via

$$(\lambda_0, \lambda_1, \lambda_2, \lambda_3) \cdot (x_0 : x_1 : x_2 : x_3) = (\lambda_0 x_0 : \lambda_1 x_1 : \lambda_2 x_2 : \lambda_3 x_3),$$

and this extends to an action upon $\tilde{X}_A$. Note that the action of the subgroup J_{F_A} is trivial. Moreover one may show (see [ABS, Prop. 1]) that $\mathrm{SL}(F_A)$ is precisely the subgroup of elements in $\mathrm{Aut}(F_A)$ that act *symplectically* upon $\tilde{X}_A$, i.e., that act trivially on the canonical bundle of $\tilde{X}_A$. Hence the group G, which satisfies $J_{F_A} \subseteq G \subseteq \mathrm{SL}(F_A)$, also acts symplectically on $\tilde{X}_A$. Since $p \nmid d$, the minimal resolution $Z_{A,G}$ of the quotient surface $\tilde{X}_A/G$ will again be a K3 surface; see [Got] for more details.

Hence, under certain conditions, a BHK pair (A, G) gives rise to the corresponding geometric object $Z_{A,G}$. We summarize all of this with the following:

Definition 3.1. *Let (A, G) be a BHK pair such that $\gcd(q_i, q_j, q_\ell) = 1$ for any three distinct weights in $\mathbf{q}$, $p \nmid q_i$ for all i, and $p \nmid d$. If the hypersurface X_A in $\mathbb{P}^3(\mathbf{q})$ is quasi-smooth and well-formed, we will call (A, G) an adequate BHK pair. Furthermore, if $Z_{A,G}$ denotes the K3 surface obtained as the minimal resolution of $\tilde{X}_A/G$, then $Z_{A,G}$ is called a K3 surface of BHK type.*

Lemma 3.2. *Suppose that the BHK pair (A, G) is an adequate BHK pair over a field of characteristic $p > 0$. Then the same is true over a field of characteristic 0.*

Proof. Put briefly, this follows from the fact that "well-formed" and "quasi-smooth" are open conditions. To see this, notice that $F_A \in \mathbb{Z}[x_0, x_1, x_2, x_3]$ cuts out a codimension one subscheme $\mathfrak{X}'_A \subseteq \mathrm{Spec}\,\mathbb{Z}[x_0, x_1, x_2, x_3]$. Likewise, if $\mathbb{Z}[x_0, x_1, x_2, x_3]$ is graded so that each x_i has weight q_i, then F_A cuts out the codimension one subscheme $\mathfrak{X}_A \subseteq \mathbb{P}^3_{\mathbb{Z}}(\mathbf{q}) = \mathrm{Proj}\,\mathbb{Z}[x_0, x_1, x_2, x_3]$. Note that $\mathfrak{X}'_A$ is the affine cone over $\mathfrak{X}_A$.

Now assume that (A, G) is adequate over a field of characteristic $p > 0$, and hence gives rise to the K3 surface $Z_{A,G}$ over this field. Let $\mathfrak{Y} \subseteq \mathfrak{X}'_A$ denote the singular locus of $\mathfrak{X}'_A$. The fact that $Z_{A,G}$ is quasi-smooth means that the fiber of $\mathfrak{Y} \to \mathrm{Spec}\,\mathbb{Z}$ over p is zero-dimensional. Hence by semi-continuity, the same is true of the generic fiber; in other words, (A, G) defines a quasi-smooth surface in characteristic 0.

Likewise, if $\mathfrak{Z} \subseteq \mathfrak{X}_A$ denotes the intersection of $\mathfrak{X}_A$ with the singular locus of $\mathbb{P}^3_{\mathbb{Z}}(\mathbf{q})$, the hypothesis of the lemma implies that the fiber of $\mathfrak{Z} \to \mathrm{Spec}\,\mathbb{Z}$ over p is either zero-dimensional or empty, and so the same must be true of the generic fiber. Hence (A, G) defines a well-formed surface in characteristic zero as well. $\square$

Remark 3.3. While they will not play a significant role in the proof of Theorem 8.2, there are two other facets of an adequate BHK pair (A, G) worth noting here.

First, results of Kreuzer and Skarke [KS, Theorem 1] classify those weighted Delsarte matrices A such that the corresponding hypersurface X_A will be quasi-smooth in characteristic 0. Their classification shows X_A is quasi-smooth if and only if the polynomial F_A is a direct sum of so-called *atomic types*, of which there are three:

Fermat: y^e

Loop: $y_0^{e_0} y_1 + y_1^{e_1} y_2 + \cdots + y_k^{e_k} y_0$

Chain: $y_0^{e_0} y_1 + y_1^{e_1} y_2 + \cdots + y_{k-1}^{e_{k-1}} y_k + y_k^{e_k}$

Moreover, one may show that their results also hold in characteristic p under the assumption that $p \nmid \det(A)$ (or equivalently, by Lemma 2.4, that $p \nmid d$): If A defines a quasi-smooth hypersurface X_A in characteristic p, then the proof of Lemma 3.2 shows the same is true in characteristic 0, and therefore F_A must be a direct sum of atomic types; for the converse, it is enough to assume that F_A consists of a single atomic type, and then one checks quasi-smoothness for each of the Fermat, loop, and chain cases under the assumption that $p \nmid \det(A)$.

The second point is that, given the assumptions made about the weight system $\mathbf{q} = (q_0, q_1, q_2, q_3)$ at the beginning of this section, $\mathbf{q}$ will belong to one of 95 possible weight systems (counted up to reordering the q_i); see [Rei, Yon]. Taken together with the Kreuzer–Skarke classification, this significantly restricts the possibilities for the matrix A in an adequate BHK pair. (On the other hand, their number is still quite large, as demonstrated by the partial lists of examples in [ABS, CLPS, CP].)

4. BHK MIRRORS

Let (A, G) be an adequate BHK pair. Since X_A is then quasi-smooth, the classification result of Kreuzer and Skarke mentioned in Remark 3.3 implies that its transpose A^T is also a weighted Delsarte matrix. Furthermore, Remark 2.3 shows that A^T also satisfies the Calabi-Yau requirement. Just as before, one obtains from A^T a polynomial F_{A^T}, which will be quasihomogeneous of some degree h_T and weight system $\mathbf{q}_T$. (We note that, in general, the degree and weight system of F_{A^T} will differ from that of F_A.) Just as with F_A, one may define the groups

$$J_{F_{A^T}} \subseteq \mathrm{SL}(F_{A^T}) \subseteq \mathrm{Aut}(F_{A^T}).$$

The group $\mathrm{Aut}(F_{A^T})$ is a subgroup of μ_d^4 (for the same value of d as in §2) and again we have $\# \mathrm{Aut}(F_{A^T}) = |\det(A)|$. Alternatively, in the additive notation, we may view $\mathrm{Aut}(F_{A^T})$ as the following subgroup of $(\mathbb{Z}/d)^4$:

$$\mathrm{Aut}(F_{A^T}) = \left\{ \mathbf{a} \in (\mathbb{Z}/d)^4 \mid \mathbf{a}A = (A^T \mathbf{a}^T)^T = 0 \right\}$$

Let us now use the additive notation to define a pairing

$$\langle \cdot, \cdot \rangle : \mathrm{Aut}(F_{A^T}) \times \mathrm{Aut}(F_A) \to \mathbb{Z}/d^2.$$

This pairing is defined by the rule

$$\langle \mathbf{a}, \mathbf{b} \rangle = \tilde{\mathbf{a}} A \tilde{\mathbf{b}}^T,$$

where $\tilde{\mathbf{a}}, \tilde{\mathbf{b}}$ are any lifts of $\mathbf{a}, \mathbf{b}$ to the group $(\mathbb{Z}/d^2)^4$; one may verify that $\langle \mathbf{a}, \mathbf{b} \rangle$ is independent of the choices of these lifts, by using the fact that $\mathbf{a}A = 0$ and $A\mathbf{b}^T = 0$

in $(\mathbb{Z}/d)^4$. This pairing is $\mathbb{Z}/d^2$-bilinear, if one regards $g \in \mathbb{Z}/d^2$ as acting upon each of the factors $\mathrm{Aut}(F_A)$ and $\mathrm{Aut}(F_{A^T})$ via scalar multiplication after applying the reduction map $\mathbb{Z}/d^2 \to \mathbb{Z}/d$.

From the adequate BHK pair (A, G) above, one may now define the *dual group* G^T as

$$G^T = \{\mathbf{a} \in \mathrm{Aut}(F_{A^T}) \mid \langle \mathbf{a}, \mathbf{b} \rangle = 0 \text{ for all } \mathbf{b} \in G\}.$$

One may check that this is equivalent to the definition of G^T in [Kel2]. (Also see [ABS, §3] for a discussion of several equivalent definitions of G^T.) Some useful facts about this construction are that $(G^T)^T = G$, $\mathrm{Aut}(F_A)^T = \{0\}$, $J_{F_A}^T = \mathrm{SL}(F_{A^T})$, and $J_{F_{A^T}} \subseteq G^T \subseteq \mathrm{SL}(F_{A^T})$. (Note that the latter two facts require the Calabi-Yau assumption and the fact that $J_{F_A} \subseteq G \subseteq \mathrm{SL}(F_A)$.)

From the discussion above, the pair (A^T, G^T) will automatically be a BHK pair, and one may also deduce (using the Kreuzer–Skarke classification) that X_{A^T} will be quasi-smooth. Hence (A^T, G^T) will be an adequate BHK pair if and only if $p \nmid q_{i,T}$ (where $\mathbf{q}_T = (q_{0,T}, q_{1,T}, q_{2,T}, q_{3,T})$) and X_{A^T} is also well-formed.

Definition 4.1. *Let (A, G) be an adequate BHK pair. If the BHK pair (A^T, G^T) is also an adequate BHK pair, then we will call it the* BHK mirror pair *of (A, G). Furthermore, we will say that the K3 surface Z_{A^T, G^T} is the* BHK mirror surface *of the surface $Z_{A,G}$.*

5. Picard numbers of K3 surfaces of BHK type

Let (A, G) be a BHK pair. We will define several subsets of the additive group $(\mathbb{Z}/d)^4$:

$$(5.1) \qquad M_d = \left\{ \mathbf{a} = (a_0, a_1, a_2, a_3) \in (\mathbb{Z}/d)^4 \ \Big| \ \sum_{i=0}^{3} a_i = 0 \right\}$$

$$(5.2) \qquad \mathfrak{A}_d = \{\mathbf{a} \in M_d \mid a_i \neq 0 \text{ for all } 0 \leq i \leq 3\}$$

Then M_d is a subgroup of $(\mathbb{Z}/d)^4$, while $\mathfrak{A}_d$ is merely a subset that is closed under scalar multiplication by $(\mathbb{Z}/d)^\times$. We define the following function upon $\mathfrak{A}_d$:

Definition 5.1. *The* age *of $\mathbf{a} = (a_0, a_1, a_2, a_3) \in \mathfrak{A}_d$ is the integer*

$$S(\mathbf{a}) = \sum_{i=0}^{3} \left\langle \frac{\hat{a}_i}{d} \right\rangle,$$

where $\hat{a}_i \in \mathbb{Z}$ is any lift of a_i from $\mathbb{Z}/d$ to $\mathbb{Z}$, and $\langle x \rangle = x - \lfloor x \rfloor$ denotes the fractional part of $x \in \mathbb{R}$.

Remark 5.2. In the literature surrounding BHK mirror symmetry, the concept of age is typically defined for any element of $\mathrm{Aut}(F_A)$ (and is not necessarily an integer). The notion is also useful in other contexts (for instance, see [IR]). For our purposes, we will only need to consider it for elements in $\mathrm{Aut}(F_A) \cap \mathfrak{A}_d$.

The following properties of $S(\mathbf{a})$ are straightforward to verify:

Proposition 5.3. *For $\mathbf{a} \in \mathfrak{A}_d$ we have:*

(1) $S(\mathbf{a}) \in \{1, 2, 3\}$
(2) $S(-\mathbf{a}) = 4 - S(\mathbf{a})$

Now define

$$\mathfrak{B}_d(0) = \left\{ \mathbf{a} \in \mathfrak{A}_d \mid S(t\mathbf{a}) = 2 \text{ for all } t \in (\mathbb{Z}/d)^{\times} \right\}.$$

If $p > 0$ is a prime such that $p \nmid d$, let f denote the order of the (congruence class of) p in the multiplicative group $(\mathbb{Z}/d)^{\times}$. We also define

$$\mathfrak{B}_d(p) = \left\{ \mathbf{a} \in \mathfrak{A}_d \ \Big| \ \sum_{j=0}^{f-1} S(tp^j \mathbf{a}) = 2 \text{ for all } t \in (\mathbb{Z}/d)^{\times} \right\}.$$

Finally, for $p = 0$ or a prime $p \nmid d$, let

$$\mathfrak{I}_d(p) = \mathfrak{A}_d \setminus \mathfrak{B}_d(p).$$

The main theorem of [Kel2] is:

Theorem 5.4 (Kelly)**.** *Let (A, G) and (A^T, G^T) be BHK mirror pairs. Over a field of characteristic $p = 0$ or $p \nmid d$, the geometric Picard numbers of the BHK mirror surfaces $Z_{A,G}$ and Z_{A^T,G^T} are given by*

$$\begin{aligned}
\rho(Z_{A,G}) &= 22 - \#(\mathfrak{I}_d(p) \cap G^T) \\
\rho(Z_{A^T,G^T}) &= 22 - \#(\mathfrak{I}_d(p) \cap G)
\end{aligned}$$

6. Quotients of Fermat surfaces

In this section, all varieties will be defined over a field of characteristic 0. We will let

$$Y_d = V(X_0^d + X_1^d + X_2^d + X_3^d)$$

denote the Fermat surface of degree d in $\mathbb{P}^3$.

The group μ_d^4 acts upon Y_d by multiplication of each coordinate:

$$(\lambda_0, \lambda_1, \lambda_2, \lambda_3) \cdot (X_0 : X_1 : X_2 : X_3) = (\lambda_0 X_0 : \lambda_1 X_1 : \lambda_2 X_2 : \lambda_3 X_3).$$

Letting $\Delta \hookrightarrow \mu_d^4$ denote the diagonal subgroup, we obtain in this way an injection $\mu_d^4 / \Delta \hookrightarrow \mathrm{Aut}(Y_d)$. By functoriality we obtain a representation of μ_d^4 / Δ upon the singular cohomology group $H^2(Y_d, \mathbb{Q})$.

We note that the group $\mathrm{Hom}(\mu_d^4 / \Delta, \mathbb{C}^{\times})$ of characters of μ_d^4 / Δ may be identified with the group M_d defined in (5.1). In particular,

$$\mathbf{a} = (a_0, a_1, a_2, a_3) \in M_d \quad \Longrightarrow \quad \mathbf{a}\big((\lambda_0, \lambda_1, \lambda_2, \lambda_3)\Delta\big) = \prod_i \lambda_i^{a_i}.$$

Hence upon extending scalars to $\mathbb{C}$, we obtain a decomposition of the middle singular cohomology of Y_d:

$$H^2(Y_d, \mathbb{C}) = \bigoplus_{\mathbf{a} \in M_d} V(\mathbf{a}),$$

where μ_d^4/Δ acts via the character $\mathbf{a}$ on $V(\mathbf{a})$. Recalling the subset $\mathfrak{A}_d \subseteq M_d$ in (5.2), one can show (see [Shi1], for instance) that

$$\dim_{\mathbb{C}} V(\mathbf{a}) = \begin{cases} 1 & \text{if } \mathbf{a} = 0 \text{ or } \mathbf{a} \in \mathfrak{A}_d \\ 0 & \text{otherwise,} \end{cases}$$

so that $H^2(Y_d, \mathbb{C})$ becomes the following direct sum of one-dimensional spaces:

$$H^2(Y_d, \mathbb{C}) = V(0) \oplus \bigoplus_{\mathbf{a} \in \mathfrak{A}_d} V(\mathbf{a}).$$

Moreover, as this decomposition arises from a group of automorphisms of Y_d, each of the subspaces $V(\mathbf{a})$ is a Hodge structure (over $\mathbb{C}$) and one can ask about its Hodge type.

Proposition 6.1. *If $\mathbf{a} \in \mathfrak{A}_d$, then $V(\mathbf{a}) \subseteq H^{2-q,q}(Y_d, \mathbb{C})$ if and only if $S(\mathbf{a}) = q+1$. Furthermore $V(0) \subseteq H^{1,1}(Y_d, \mathbb{C})$.*

Proof. See [Shi1]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Now let (A, G) be an adequate BHK pair. In [Kel1, Cor. 3.2], the K3 surface $Z_{A,G}$ is shown to be birational to the quotient Y_d/Γ, for a particular subgroup $\Gamma \subseteq \mu_d^4$ (whose precise definition will not be needed here). Thus we have an isomorphism of Hodge structures

$$H^2(Z_{A,G}, \mathbb{Q}) \simeq \mathbf{E} \oplus H^2(Y_d, \mathbb{Q})^{\Gamma},$$

where $\mathbf{E}$ is generated by the classes of some (-1)-curves on $Z_{A,G}$. Tensoring with $\mathbb{C}$, we obtain

$$\begin{aligned} H^2(Z_{A,G}, \mathbb{C}) &\simeq & \mathbf{E}_{\mathbb{C}} \oplus H^2(Y_d, \mathbb{C})^{\Gamma} \\ &=& \mathbf{E}_{\mathbb{C}} \oplus \left(V(0) \oplus \bigoplus_{\mathbf{a} \in \mathfrak{A}_d} V(\mathbf{a}) \right)^{\Gamma} \\ &=& \mathbf{E}_{\mathbb{C}} \oplus V(0) \oplus \bigoplus_{\mathbf{a} \in \mathfrak{A}_d \cap L(\Gamma)} V(\mathbf{a}), \end{aligned}$$

where

$$L(\Gamma) = \{ \mathbf{a} \in M_d \mid \mathbf{a}(\gamma) = 1 \text{ for all } \gamma \in \Gamma \}.$$

But in [Kel2, Prop. 3], Kelly shows that

$$L(\Gamma) = G^T \cap M_d,$$

where G^T is the dual group of the group G. Since $\mathfrak{A}_d \subseteq M_d$, we arrive at

$$H^2(Z_{A,G}, \mathbb{C}) \simeq \mathbf{E}_{\mathbb{C}} \oplus V(0) \oplus \bigoplus_{\mathbf{a} \in \mathfrak{A}_d \cap G^T} V(\mathbf{a}).$$

By Proposition 6.1, this also gives the following description of the Hodge decomposition of $H^2(Z_{A,G}, \mathbb{C})$:

$$(6.1) \qquad H^{2,0}(Z_{A,G}, \mathbb{C}) \quad \simeq \quad \bigoplus_{\substack{\mathbf{a} \in \mathfrak{A}_d \cap G^T \\ S(\mathbf{a})=1}} V(\mathbf{a})$$

$$(6.2) \qquad H^{1,1}(Z_{A,G}, \mathbb{C}) \quad \simeq \quad \mathbf{E}_{\mathbb{C}} \oplus V(0) \oplus \bigoplus_{\substack{\mathbf{a} \in \mathfrak{A}_d \cap G^T \\ S(\mathbf{a})=2}} V(\mathbf{a})$$

$$(6.3) \qquad H^{0,2}(Z_{A,G}, \mathbb{C}) \quad \simeq \quad \bigoplus_{\substack{\mathbf{a} \in \mathfrak{A}_d \cap G^T \\ S(\mathbf{a})=3}} V(\mathbf{a})$$

Proposition 6.2. *Let (A, G) and (A^T, G^T) be BHK mirror pairs. Each of the sets $\mathfrak{A}_d \cap G$ and $\mathfrak{A}_d \cap G^T$ contains exactly one element of age 1.*

Proof. Since Z_{A_G} is a K3 surface, we know that $\dim H^{2,0}(Z_{A,G}, \mathbb{C}) = 1$. Hence by (6.1), there is exactly one $\mathbf{a} \in \mathfrak{A}_d \cap G^T$ such that $S(\mathbf{a}) = 1$. By symmetric reasoning we have

$$H^{2,0}(Z_{A^T, G^T}, \mathbb{C}) \simeq \bigoplus_{\substack{\mathbf{a} \in \mathfrak{A}_d \cap G \\ S(\mathbf{a})=1}} V(\mathbf{a}),$$

from which we conclude that $\mathfrak{A}_d \cap G$ also has exactly one element of age 1. $\qquad \square$

7. Description in terms of orbits

The results in this section are of a strictly combinatorially nature. Specifically, we will work with an integer $d \geq 1$, a subgroup $H \subseteq (\mathbb{Z}/d)^4$, and primes p that do not divide d. Our goal is to analyze the structure of the sets $\mathfrak{I}_d(0) \cap H$ and $\mathfrak{I}_d(p) \cap H$ under an assumption on H inspired by Proposition 6.2. The results obtained will be applied to the geometric setting in the next section.

We let $U_d = (\mathbb{Z}/d)^\times$. This group acts upon $(\mathbb{Z}/d)^4$ by multiplication, and it preserves the subset $\mathfrak{A}_d$ and any subgroup of $(\mathbb{Z}/d)^4$. In this section, we fix some subgroup $H \subseteq (\mathbb{Z}/d)^4$. Hence U_d acts upon the set $\mathfrak{A}_d \cap H$, and we let

$$\Phi_H = (\mathfrak{A}_d \cap H)/U_d$$

denote the collection of orbits of this action. For a prime $p \nmid d$, let $\langle p \rangle \subseteq U_d$ denote the subgroup generated by (the congruence class of) p. Since the subgroup $\langle p \rangle$ will of course preserve any orbit $\mathcal{O} \in \Phi_H$ of the larger group U_d, we make another piece of notation: Let

$$\Phi_H(p, \mathcal{O}) = \mathcal{O}/\langle p \rangle$$

denote the collection of orbits of $\langle p \rangle$ acting upon a U_d-orbit $\mathcal{O}$. Thus we have decomposed each $\mathcal{O} \in \Phi_H$ as the disjoint union

$$\mathcal{O} = \coprod_{T \in \Phi_H(p, \mathcal{O})} T$$

and decomposed $\mathfrak{A}_d \cap H$ as

$$\mathfrak{A}_d \cap H = \coprod_{\mathcal{O} \in \Phi_H} \mathcal{O} = \coprod_{\mathcal{O} \in \Phi_H} \left(\coprod_{T \in \Phi_H(p, \mathcal{O})} T \right).$$

We now restate the definitions of the sets $\mathfrak{B}_d(0) \cap H$ and $\mathfrak{I}_d(0) \cap H$ in terms of the orbits Φ_H. First note that if $\mathbf{a} \in \mathfrak{A}_d \cap H$, then

$$\mathbf{a} \in \mathfrak{B}_d(0) \cap H \iff S(t\mathbf{a}) = 2 \text{ for all } t \in U_d$$
$$\iff S(\mathbf{b}) = 2 \text{ for all } \mathbf{b} \in U_d \cdot \mathbf{a},$$

where $U_d \cdot \mathbf{a}$ denotes the U_d-orbit of $\mathbf{a}$. Therefore $\mathfrak{B}_d(0) \cap H$ is a union of certain orbits in Φ, namely those $\mathcal{O} \in \Phi_H$ that only contain elements of age 2, i.e,

$$\mathfrak{B}_d(0) \cap H = \coprod \{\mathcal{O} \in \Phi_H \mid S(\mathbf{a}) = 2 \text{ for all } \mathbf{a} \in \mathcal{O}\}.$$

Therefore, taking the complement of $\mathfrak{B}_d(0) \cap H$ inside $\mathfrak{A}_d \cap H$, we obtain

$$\mathfrak{I}_d(0) \cap H = \coprod \{\mathcal{O} \in \Phi_H \mid S(\mathbf{a}) \neq 2 \text{ for some } \mathbf{a} \in \mathcal{O}\}.$$

But if $S(\mathbf{a}) \neq 2$, then by Proposition 5.3, either $S(\mathbf{a}) = 1$ or $S(-\mathbf{a}) = 1$. Therefore, we may alternatively write

(7.1) $$\mathfrak{I}_d(0) \cap H = \coprod \{\mathcal{O} \in \Phi_H \mid S(\mathbf{a}) = 1 \text{ for some } \mathbf{a} \in \mathcal{O}\}.$$

In words, this says that $\mathfrak{I}_d(0) \cap H$ is the union of those U_d-orbits $\mathcal{O}$ that contain at least one element of age 1.

From (7.1), we can deduce:

Proposition 7.1. *Suppose that $\mathfrak{A}_d \cap H$ contains exactly one element $\mathbf{a}_0$ of age 1. If h denotes the order of $\mathbf{a}_0$ in $(\mathbb{Z}/d)^4$, then*

$$\mathfrak{I}_d(0) \cap H = \{n\mathbf{a}_0 \mid 1 \leq n \leq h, (n, h) = 1\}$$

is a set of cardinality $\phi(h)$, where ϕ denotes Euler's totient function.

Proof. By (7.1), $\mathfrak{I}_d(0) \cap H$ is equal to the orbit of $\mathbf{a}_0$ under U_d. That is, $\mathfrak{I}_d(0) \cap H = R_d$ where

$$R_d = \{m\mathbf{a}_0 \mid 1 \leq m \leq d, (m, d) = 1\}.$$

Letting $R_h = \{n\mathbf{a}_0 \mid 1 \leq n \leq h, (n, h) = 1\}$, we must show that $R_d = R_h$.

If $m\mathbf{a}_0 \in R_d$, let $m \equiv m_0 \pmod{h}$, where $1 \leq m_0 \leq h$. Since $h \mid d$, we have $(m_0, h) = 1$, and so $m\mathbf{a}_0 = m_0\mathbf{a}_0 \in R_h$. Thus $R_d \subseteq R_h$.

For the opposite inclusion, take $n\mathbf{a}_0 \in R_h$. The main obstacle to concluding that $n\mathbf{a}_0 \in R_d$ is that we might have $(n, h) = 1$ but $(n, d) > 1$. If the primes dividing h and d are identical, this will not happen; so assuming otherwise, suppose that $Q > 1$ denotes the product of all primes that divide d but not h. By the Chinese Remainder Theorem, there exists an integer N satisfying $1 \leq N \leq hQ \leq d$, $N \equiv n \pmod{h}$ and $N \equiv 1 \pmod{Q}$. Then N will be not be divisible by any primes dividing d, and thus $n\mathbf{a}_0 = N\mathbf{a}_0 \in R_d$.

Finally, we note that all elements of R_h are distinct since the order of $\mathbf{a}_0$ is h, so it has cardinality $\phi(h)$. $\square$

Likewise, we may recast $\mathfrak{B}_d(p) \cap H$ and $\mathfrak{I}_d(p) \cap H$ in terms of the orbits $\Phi_H(p)$. Recalling that f is the order of p in U_d, we note that if $\mathbf{a} \in \mathfrak{A}_d \cap H$ then

$$\mathbf{a} \in \mathfrak{B}_d(p) \cap H \iff \sum_{j=0}^{f-1} S(tp^j\mathbf{a}) = 2f \text{ for all } t \in U_d$$

$$\iff \sum_{j=0}^{f-1} S(p^j\mathbf{b}) = 2f \text{ for all } \mathbf{b} \in U_d \cdot \mathbf{a}.$$

Now the sum $\sum_{j=0}^{f-1} S(p^j\mathbf{b})$ consists of f terms, so we have

$$\sum_{j=0}^{f-1} S(p^j\mathbf{b}) = 2f \iff \sum_{j=0}^{f-1} (S(p^j\mathbf{b}) - 2) = 0.$$

The quantity $S(p^j\mathbf{b}) - 2$ takes values in $\{-1, 0, 1\}$ by Proposition 5.3, and so the equation on the right side is true if and only if there are an equal number of occurrences of 1 and -1 in the sum. That is, we have $\sum_{j=0}^{f-1} S(p^j\mathbf{b}) = 2f$ if and only if 1 and 3 appear in equal numbers in the sequence

$$S(\mathbf{b}), S(p\mathbf{b}), \dots, S(p^{f-1}\mathbf{b}).$$

Finally, we note that sequence

(7.2) $$\mathbf{b}, p\mathbf{b}, \dots, p^{f-1}\mathbf{b}$$

is simply the orbit of $\mathbf{b}$ under $\langle p \rangle$, but possibly repeated a certain number of times. (More precisely, if the $\langle p \rangle$-orbit of $\mathbf{b}$ has size f', then $f' \mid f$ and the sequence (7.2) repeats this orbit f/f' times.) Hence we conclude

$$\mathbf{a} \in \mathfrak{B}_d(p) \cap H \iff \sum_{j=0}^{f-1} S(p^j\mathbf{b}) = 2f \text{ for all } \mathbf{b} \in U_d \cdot \mathbf{a}$$

$$\iff \#\big((\langle p \rangle \cdot \mathbf{b}) \cap S^{-1}(1)\big) = \#\big((\langle p \rangle \cdot \mathbf{b}) \cap S^{-1}(3)\big) \text{ for all } \mathbf{b} \in U_d \cdot \mathbf{a}$$

As with $\mathfrak{B}_d(0) \cap H$, we see that $\mathfrak{B}_d(p) \cap H$ is a union of certain orbits in Φ_H. Looking at the preceding equivalence, we can state this precisely as:

$$\mathfrak{B}_d(p) \cap H = \coprod \left\{ \mathcal{O} \in \Phi_H \mid \#(T \cap S^{-1}(1)) = \#(T \cap S^{-1}(3)) \text{ for all } T \in \Phi_H(p, \mathcal{O}) \right\}.$$

Therefore we have

(7.3)
$$\mathfrak{I}_d(p) \cap H = \coprod \left\{ \mathcal{O} \in \Phi_H \mid \#(T \cap S^{-1}(1)) \neq \#(T \cap S^{-1}(3)) \text{ for some } T \in \Phi_H(p, \mathcal{O}) \right\}.$$

In words, $\mathfrak{I}_d(p) \cap H$ is the union of those U_d-orbits $\mathcal{O}$ that contain at least one $\langle p \rangle$-orbit T having unequal numbers of elements of age 1 and 3.

Proposition 7.2. *Suppose that $\mathfrak{A}_d \cap H$ contains exactly one element $\mathbf{a}_0$ of age 1. If h denotes the order of $\mathbf{a}_0$ in $(\mathbb{Z}/d)^4$, then we have the following two possibilities for $\mathfrak{I}_d(p) \cap H$:*

(1) If $p^\ell \equiv -1 \pmod{h}$ for some ℓ, then $\mathfrak{I}_d(p) \cap H = \emptyset$.
(2) If $p^\ell \not\equiv -1 \pmod{h}$ for all ℓ, then $\mathfrak{I}_d(p) \cap H = \mathfrak{I}_d(0) \cap H$.

Proof. By Proposition 5.3, the hypothesis implies that $-\mathbf{a}_0$ is the only element of $\mathfrak{A}_d \cap H$ of age 3. Hence $U_d \cdot \mathbf{a}_0$ is the only orbit in Φ_H that doesn't consist entirely of elements of age 2. So by (7.3), $\mathfrak{I}_d(p) \cap H$ can only be empty or equal to $U_d \cdot \mathbf{a}_0$, which in turn is equal to $\mathfrak{I}_d(0) \cap H$ by Proposition 7.1.

If we assume that $p^\ell \equiv -1 \pmod{h}$, we find that $\langle p \rangle \cdot \mathbf{a}_0$ contains one element of age 1 and one element of age 3, and this gives the first statement. But assuming $p^\ell \not\equiv -1 \pmod{h}$ for all k, it follows that the $\langle p \rangle \cdot \mathbf{a}_0$ contains one element of age 1 and no elements of age 3; hence $U_d \cdot \mathbf{a}_0$ is contained in $\mathfrak{I}_d(p) \cap H$, giving the second statement. $\qquad\square$

8. Picard numbers of K3 surfaces of BHK type

Let (A, G) be an adequate BHK pair, and let d be defined as in (2.6). As before, this defines a polynomial F_A that is quasihomogeneous of degree h with weight system $\mathbf{q} = (q_0, q_1, q_2, q_3)$. By assumption, the group G satisfies $J_{F_A} \subseteq G \subseteq \mathrm{SL}(F_A)$, and thus G contains the element

$$\mathbf{j}_A = \frac{d}{h}\mathbf{q} = \left(\frac{q_0 d}{h}, \frac{q_1 d}{h}, \frac{q_2 d}{h}, \frac{q_3 d}{h} \right),$$

which is a generator of J_{F_A}. Now

$$(8.1) \qquad S(\mathbf{j}_A) = \sum_{i=0}^{3} \left\langle \frac{q_i d/h}{d} \right\rangle = \sum_{i=0}^{3} \left\langle \frac{q_i}{h} \right\rangle = \sum_{i=0}^{3} \frac{q_i}{h} = 1$$

due to the Calabi-Yau requirement $\sum_i q_i = h$.

Lemma 8.1. *Let (A, G) and (A^T, G^T) be BHK mirror pairs. Then $\mathbf{j}_A$ is the only element of age 1 in the set $\mathfrak{A}_d \cap G^T$, where G^T is the dual group of G.*

Proof. By Lemma 3.2, we know that (A, G) gives rise to the K3 surface in characteristic 0 that we have denoted by $Z_{A,G}$. Applying (8.1) and Proposition 6.2 to this surface, we obtain the desired conclusion. $\qquad\square$

Our main result now follows:

Theorem 8.2. *Let (A, G) be an adequate BHK pair and (A^T, G^T) its BHK mirror pair. Let h (resp. h_T) denote the degree of the quasihomogeneous polynomial F_A (resp. F_{A^T}). Over the algebraically close field k, the Picard numbers of the BHK mirror surfaces $Z_{A,G}$ and Z_{A^T,G^T} are given as follows:*

(1) *If* $\operatorname{char} k = 0$ *then*

$$\begin{aligned} \rho(Z_{A,G}) &= 22 - \phi(h_T) \\ \rho(Z_{A^T,G^T}) &= 22 - \phi(h). \end{aligned}$$

(2) *If* $\operatorname{char} k = p > 0$ *with* $p \nmid d$ *then*

$$\rho(Z_{A,G}) = \begin{cases} 22 & \text{if } p^\ell \equiv -1 \pmod{h_T} \text{ for some } \ell \\ 22 - \phi(h_T) & \text{if } p^\ell \not\equiv -1 \pmod{h_T} \text{ for all } \ell \end{cases}$$

$$\rho(Z_{A^T,G^T}) = \begin{cases} 22 & \text{if } p^\ell \equiv -1 \pmod{h} \text{ for some } \ell \\ 22 - \phi(h) & \text{if } p^\ell \not\equiv -1 \pmod{h} \text{ for all } \ell \end{cases}$$

Proof. Applying Lemma 8.1 to (A^T, G^T), we find that $\mathbf{j}_A$ is the only element of age 1 in G. Now the order of $\mathbf{j}_A$ in $(\mathbb{Z}/d)^4$ clearly divides h, and by using the fact that the weights q_i have no nontrivial common divisor, one may deduce that its order is exactly h. Therefore we conclude by Proposition 7.1 that

$$(8.2) \qquad \mathfrak{I}_d(0) \cap G = \{n\mathbf{j}_A \mid 1 \leq n \leq h, (n, h) = 1\}$$

has cardinality $\phi(h)$. We may apply a symmetric line of reasoning to describe the structure of $\mathfrak{I}_d(0) \cap G^T$. Combining this with Theorem 5.4 and Proposition 7.2, we obtain the theorem. $\qquad \square$

Remark 8.3. In [Ino, Cor. 1.2], Inose proves that if there is a rational map of finite degree between two complex K3 surfaces, then the Picard numbers of both K3 surfaces are equal; this implies that, in characteristic 0, the value of $\rho(Z_{A,G})$ depends only upon A and not upon G. By invoking this fact, one could reduce the proof of part (1) of Theorem 8.2 to just proving that $\#(\mathfrak{I}_d(0) \cap J_{F_A}) = \phi(h)$, and this in turn (by Proposition 7.1) would follow if one could establish that $\mathfrak{A}_d \cap J_{F_A}$ contains exactly one element of age 1.

Given that $J_{F_A} = \langle \mathbf{j}_A \rangle$ is determined only by the weight system $\mathbf{q}$, one might hope to prove this latter statement in a more direct, combinatorial fashion that avoids the use of Hodge theory (which was invoked in Proposition 6.2). However, such an approach appears to be fairly subtle. For instance, the statement does not follow from the Calabi-Yau condition alone (see Definition (2.2)), as the example of $\mathbf{q} = (4, 1, 1, 1)$, $h = 7$ demonstrates: Here $\mathfrak{A}_d \cap J_{F_A}$ contains two elements of age 1. One possible reason for this may be related to the fact that hypersurfaces of degree 7 in $\mathbb{P}^3(4, 1, 1, 1)$ are not quasi-smooth (see [IF, Cor. 8.5]). Thus, in order to prove the statement at the end of the previous paragraph, one might also need to incorporate an assumption guaranteeing the quasi-smoothness of hypersurfaces of degree $\sum_i q_i$ inside $\mathbb{P}^3(\mathbf{q})$. But incorporating quasi-smoothness, well-formedness, and such desirable assumptions imposes several combinatorial requirements [IF] on $\mathbf{q}$ that do not indicate a clear path forward; in brief, the use of Hodge theory seems more direct.

9. EXAMPLES

We now present some examples. The first example is drawn from [Kel2], and the next two discuss cases of classical interest (quartic surfaces and double sextics). The final example demonstrates Corollary 1.4 by giving one instance of a BHK surface with each possible Picard number.

In the first example, we consider both positive characteristic and characteristic 0, but in the later ones we only concern ourselves with characteristic 0; the interested reader may apply part (2) of Theorem 8.2 to figure out the Picard number in positive characteristic for these cases.

Example 9.1. This example is taken from §4 of [Kel2]. Let

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 6 & 1 \\ 0 & 0 & 0 & 7 \end{bmatrix},$$

so that

$$F_A = x_0^2 x_1 + x_1^2 x_2 + x_2^6 x_3 + x_3^7,$$

which is quasihomogeneous of degree $h = 7$ and weights $\mathbf{q} = (2, 3, 1, 1)$. Thus $\mathrm{Aut}(F_A)$ is a group of order $|\det(A)| = 168$, and one may show (see [Kel2]) that $\mathrm{SL}(F_A)$ has order 21. Since $J_{F_A} = \langle \mathbf{j}_A \rangle$ has order $h = 7$, it follows that $[\mathrm{SL}(F_A) : J_{F_A}] = 3$, and thus there are only two groups G such that $J_{F_A} \subseteq G \subseteq \mathrm{SL}(F_A)$, namely $G = J_{F_A}$ and $G = \mathrm{SL}(F_A)$.

Reading off the rows of A^T, we also have

$$F_{A^T} = x_0^2 + x_0 x_1^2 + x_1 x_2^6 + x_2 x_3^7,$$

which is quasihomogeneous of degree $h_T = 8$ and weights $\mathbf{q_T} = (4, 2, 1, 1)$. In this case, $J_{F_{A^T}}$ is cyclic of order 8, $\mathrm{SL}(F_{A^T})$ is cyclic of order 24, and so, depending upon G, one either has $G^T = J_{F_{A^T}}$ or $G^T = \mathrm{SL}(F_{A^T})$. The precise correspondence happens to be

$$(J_{F_A})^T = \mathrm{SL}(F_{A^T}), \qquad SL(F_A)^T = J_{F_{A^T}},$$

although for the purposes of computing Picard numbers, Theorem 8.2 says that the choice of G is irrelevant. If $Z_{A,G}$ and Z_{A^T,G^T} are taken to be defined over k of characteristic zero, then the first part of the theorem gives

$$\rho(Z_{A,G}) = 22 - \phi(h_T) = 22 - \phi(8) = 18$$

and

$$\rho(Z_{A^T,G^T}) = 22 - \phi(h) = 22 - \phi(7) = 16.$$

Upon computing

$$A^{-1} = \begin{bmatrix} 1/2 & -1/4 & 1/24 & -1/168 \\ 0 & 1/2 & -1/12 & 1/84 \\ 0 & 0 & 1/6 & -1/42 \\ 0 & 0 & 0 & 1/7 \end{bmatrix},$$

we find that $d = 168$. If char $k = p$ and $p \nmid d$ then, in this example, the assumption $p \nmid d$ guarantees that p does not divide the weights in $\mathbf{q}$ and $\mathbf{q}_T$. We also note that $p \nmid h$ and $p \nmid h_T$, since h, h_T both divide d. Therefore $p \equiv b \pmod{h_T}$ for some $1 \le b \le h_T$ such that $(b, h_T) = 1$. Since $h_T = 8$, this gives $p \equiv 1, 3, 5,$ or $7 \pmod 8$. Only in the case $p \equiv 7 \pmod 8$ do we have $p^\ell \equiv -1 \pmod 8$ for some ℓ. Therefore by Theorem 8.2

$$\rho(Z_{A,G}) = \begin{cases} 22 & \text{if } p \equiv 7 \pmod 8 \\ 18 & \text{if } p \equiv 1, 3, 5 \pmod 8. \end{cases}$$

On the other hand, we have $p \equiv 1, 2, 3, 4, 5,$ or $6 \pmod 7$, and so $p^\ell \equiv -1 \pmod 7$ for some ℓ if and only if $p \equiv 3, 5, 6 \pmod 7$. Therefore

$$\rho(Z_{A^T, G^T}) = \begin{cases} 22 & \text{if } p \equiv 3, 5, 6 \pmod 7 \\ 16 & \text{if } p \equiv 1, 2, 4 \pmod 7. \end{cases}$$

This explains the observations in [Kel2, p.62].

Example 9.2 (Quartic surfaces of BHK type).

We suppose $p = \operatorname{char}(k) = 0$.

A smooth hypersurface in $\mathbb{P}^3$ of degree 4 is a *quartic surface*, and is in fact a K3 surface. The surface $Z_{A,G}$ arising from an adequate BHK pair (A, G) with weights $\mathbf{q} = (1, 1, 1, 1)$ and degree $h = 4$ will be the minimal resolution of a quotient of the quartic surface X_A. The main task in determining all such K3 surfaces of BHK type is to write down the possibilities for the homogeneous polynomial F_A of degree 4, and the classification of Kreuzer and Skarke (see Remark 3.3) makes this straightforward. Upon doing so, the resulting quasi-smooth surfaces X_A in $\mathbb{P}^3$ will be automatically well-formed and thus smooth (given that $\mathbb{P}^3$ is smooth). If the BHK pair (A^T, G^T) is also adequate, then Theorem 8.2 implies that $\rho(Z_{A,G})$ depends only upon A and not upon G.

Recall that the Kreuzer–Skarke classification dictates that F_A must be a sum of the three atomic types: Fermat, loop, and chain. When F_A is consists only of Fermat and loop types, the mirror polynomial F_{A^T} will be essentially the same as F_A; more precisely, F_A and F_{A^T} will be equal up to a permutation of the variables. In such a case, the surfaces X_A and X_{A^T} both live in $\mathbb{P}^3$ and are isomorphic; in particular, the degree of the mirror polynomial will be $h_T = 4$ and thus

$$\rho(Z_{A,G}) = \rho(X_A) = 22 - \phi(4) = 20$$

for any G. On the other hand, when the chain atomic type is included, more variation appears. In such cases, one may check that (A^T, G^T) is adequate (which amounts to verifying the necessary conditions on the weight system $\mathbf{q_T}$ and that X_{A^T} is well-formed) and then apply Theorem 8.2 to compute $\rho(Z_{A,G})$.

The results are summarized in Table 1; for ease of reading, the variables x, y, z, w are used in place of $x_0, \ldots, x_3$. The polynomial F_A appears in the second column, the weights (arranged in decreasing order) and degree of the mirror polynomial F_{A^T} make up the third column, and the fourth column gives $\rho(Z_{A,G}) = 22 - \phi(h_T)$. Note that the polynomials in rows 1–5 are sums of Fermat and loop atomic types, while those in rows 6–10 have at least one summand of chain type.

Remark 9.3. Polynomials F_A in distinct rows of Table 1 may in fact yield isomorphic surfaces X_A over k. As one example, the following shows that the polynomials in

TABLE 1. Picard numbers of quartic surfaces of BHK type

	F_A	$(\mathbf{q_T}; h_T)$	$\rho(Z_{A,G})$
1	$x^4 + y^4 + z^4 + w^4$	$(1,1,1,1;4)$	20
2	$x^4 + y^4 + z^3w + w^3z$	$(1,1,1,1;4)$	20
3	$x^4 + y^3z + z^3w + w^3y$	$(1,1,1,1;4)$	20
4	$x^3y + y^3x + z^3w + w^3z$	$(1,1,1,1;4)$	20
5	$x^3y + y^3z + z^3w + w^3x$	$(1,1,1,1;4)$	20
6	$x^3y + y^4 + z^3w + w^4$	$(2,2,1,1;6)$	20
7	$x^4 + y^4 + z^3w + w^4$	$(4,3,3,2;12)$	18
8	$x^3y + y^3x + z^3w + w^4$	$(4,3,3,2;12)$	18
9	$x^4 + y^3z + z^3w + w^4$	$(12,9,8,7;36)$	10
10	$x^3y + y^3z + z^3w + w^4$	$(9,7,6,5;27)$	4

rows 1 and 2 yield isomorphic surfaces:

$$
\begin{aligned}
x^4 + y^4 + z^4 + w^4 &= x^4 + y^4 + z^4 - (\zeta_8 w)^4 \\
&= x^4 + y^4 + (z^2 + (\zeta_8 w)^2)(z + \zeta_8 w)(z - \zeta_8 w) \\
&= x^4 + y^4 + \frac{1}{2}(z')^3 w' + \frac{1}{2}(w')^3 z' \\
&= x^4 + y^4 + (z'')^3 w'' + (w'')^3 z''
\end{aligned}
$$

where $\zeta_8 \in k$ is a primitive 8th root of unity, $\eta \in k$ satisfies $\eta^4 = 1/2$, and we put $z' = z + \zeta_8 w$, $w' = z - \zeta_8 w$, $z'' = \eta z'$, and $w'' = \eta w'$. By the same calculation, we see that the surfaces X_A obtained from rows 1, 2, and 4 are all isomorphic, and that the surfaces obtained from rows 7 and 8 are isomorphic.

From this, we see that the number of isomorphism classes of surfaces X_A represented in Table 1 is at most 7; on the other hand, by looking at the last column, the number of isomorphism classes is also at least 4. We have not attempted to determine this number more precisely than this.

Example 9.4 (Double sextics of BHK type).

We suppose $p = \mathrm{char}(k) = 0$.

If B is a smooth curve of degree 6 in $\mathbb{P}^2$, then the double cover Y_B of $\mathbb{P}^2$ branched over B will be a K3 surface. The surface Y_B is a *double sextic*, and one may view it as a hypersurface of degree 6 in $\mathbb{P}^3(3,1,1,1)$. The surface $Z_{A,G}$ arising from an adequate BHK pair (A,G) will be a quotient of a double sextic if it has weights $\mathbf{q} = (3,1,1,1)$ and degree $h = 6$.

To write down all polynomials F_A giving such surfaces, one again uses the Kreuzer-Skarke classification. One may write down the Fermat, loop, and chain atomic types of degree 6 having weights $(1, \ldots, 1)$ or $(3, 1, \ldots, 1)$, and then form polynomials F_A containing exactly one type with the latter weight system; for instance, $x^6 + xy^5$ is a chain with weights $(3, 1)$ and degree 6, while $z^5w + w^5z$ is a loop with weights $(1, 1)$ and degree 6, and so one example is $F_A = x^6 + xy^5 + z^5w + w^5z$ with weights $\mathbf{q} = (3, 1, 1, 1)$ and degree $h = 6$.

TABLE 2. Picard numbers of double sextics of BHK type

	F_A	$(\mathbf{q_T}; h_T)$	$\rho(Z_{A,G})$
1	$x^2 + y^6 + z^6 + w^6$	$(3, 1, 1, 1; 6)$	20
2	$x^2 + y^6 + z^5w + w^5z$	$(3, 1, 1, 1; 6)$	20
3	$x^2 + y^5z + z^5w + w^5y$	$(3, 1, 1, 1; 6)$	20
4	$x^2 + xy^3 + z^6 + w^6$	$(2, 2, 1, 1; 6)$	20
5	$x^2 + xy^3 + z^5w + w^5z$	$(2, 2, 1, 1; 6)$	20
6	$x^2 + xy^3 + z^5w + w^6$	$(5, 5, 3, 2; 15)$	14
7	$x^2 + xy^3 + yz^5 + w^6$	$(11, 8, 6, 5; 30)$	14
8	$x^2 + y^6 + z^5w + w^6$	$(15, 6, 5, 4; 30)$	14
9	$x^2 + xy^3 + yz^5 + zw^5$	$(9, 7, 5, 4; 25)$	2
10	$x^2 + y^5z + z^5w + w^6$	$(25, 10, 8, 7; 50)$	2

In this way one obtains the polynomials listed in Table 2; note that the variables have been arranged so that x has weight 3. The surface X_A defined by any one of these polynomials F_A does not contain $(1 : 0 : 0 : 0)$, the unique singular point of $\mathbb{P}^3(3, 1, 1, 1)$, and so is not only well-formed but smooth. Hence (A, G) will be an adequate BHK pair for any $J_{F_A} \subseteq G \subseteq \mathrm{SL}(F_A)$. As in Example 9.2, if F_A does not contain any chain summands then the surfaces X_A and X_{A^T} will both live in $\mathbb{P}^3(3, 1, 1, 1)$ and be isomorphic via a permutation of variables; in this case, one will have $\rho(Z_{A,G}) = 22 - \phi(6) = 20$. These polynomials make up rows 1–3 of the table. When F_A contains at least one chain summand, one may check in each case that (A^T, G^T) is adequate, and hence Theorem 8.2 applies for the calculation of $\rho(Z_{A,G})$.

Remark 9.5. As in Example 9.2, polynomials in distinct rows of Table 2 can yield isomorphic surfaces over k; this is evidenced, for instance, by the equality

$$x^2 + xy^3 \ = \ (x + \frac{1}{2}y^3)^2 - \frac{1}{4}y^6.$$

More precisely, this equality shows that, over k, the surfaces X_A coming from rows 1 and 4 are isomorphic, the surfaces from rows 2 and 5 are isomorphic, the surfaces from rows 6–8 are all isomorphic, and the surfaces from rows 9 and 10 are isomorphic. Hence the number of isomorphism classes of the surfaces X_A represented in Table 2 lies between 3 and 5.

Example 9.6.

We suppose $p = \mathrm{char}(k) = 0$.

To verify Corollary 1.4, we present in Table 3 a list of polynomials F_A that give rise to K3 surfaces of BHK type having every possible Picard number permitted by Theorem 8.2. The examples listed in Table 3 are drawn (rather arbitrarily) from those in Table 1, Table 2, and the tables in [ABS, CLPS, CP]. The first column lists F_A, the second lists the weights and degree of F_A, the third lists the weights and degree of F_{A^T}, and the fourth lists the Picard number of the surfaces $Z_{A,G}$.

TABLE 3. Examples of K3 surfaces of BHK type with each possible Picard number

F_A	$(\mathbf{q}; h)$	$(\mathbf{q_T}; h_T)$	$\rho(Z_{A,G})$
$x^2 + xy^3 + yz^5 + zw^5$	$(3, 1, 1, 1; 6)$	$(9, 7, 5, 4; 25)$	2
$x^3y + y^3z + z^3w + w^4$	$(1, 1, 1, 1; 4)$	$(9, 7, 6, 5; 27)$	4
$x^2 + y^5 + z^5 + zw^8$	$(5, 2, 2, 1; 10)$	$(20, 8, 7, 5; 40)$	6
$x^4 + y^3z + z^3w + w^4$	$(1, 1, 1, 1; 4)$	$(12, 9, 8, 7; 36)$	10
$x^2 + y^3z + yz^4 + w^{22}$	$(11, 6, 4, 1; 22)$	$(11, 6, 4, 1; 22)$	12
$x^2 + xy^3 + z^5w + w^6$	$(3, 1, 1, 1; 6)$	$(5, 5, 3, 2; 15)$	14
$x^2 + xy^2 + z^7 + yw^7$	$(14, 7, 4, 3; 28)$	$(3, 2, 1, 1; 7)$	16
$x^4 + y^4 + z^3w + w^4$	$(1, 1, 1, 1; 4)$	$(4, 3, 3, 2; 12)$	18
$x^4 + y^4 + z^4 + w^4$	$(1,1,1,1;4)$	$(1,1,1,1;4)$	20

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