# A FAMILY OF SURFACES WITH $p_g = q = 1$, $K^2 = 2$, AND LARGE PICARD NUMBER

CHRISTOPHER LYONS

Abstract. We give a one-parameter family of surfaces over $\mathbb{Q}$ with $p_g = q = 1$ and $K^2 = 2$ whose Picard number is at least 8 (and in some cases is exactly 8). This is close to the Hodge-theoretic upper bound $h^{1,1} = 10$, making them analogous and conjecturally related to a family of K3 surfaces of Picard number at least 18.

## 1. Introduction

Let $k$ be a finitely-generated field of characteristic 0, with algebraic closure $\bar{k}$, and let $X$ denote smooth projective algebraic surface over $k$. The geometric Picard number $\rho$ of $X$ is defined as a rank of the Néron-Severi group $\mathrm{NS}(X_{\bar{k}})$, where the latter denotes the group of divisors on $X_{\bar{k}}$ modulo algebraic equivalence. By Hodge Theory, one has the upper bound $\rho \leq h^{1,1} = B_2 - 2p_g$, where $B_2$ is the second Better number and $p_g$ is the geometric genus, and we may say (informally) that $X$ has large Picard number if $\rho$ is "close" to this upper bound.

When the Picard number of $X$ is large, this not only points to the existence of interesting sources of algebraic curves on $X$, but also to convenient geometric and arithmetic features. The case of K3 surfaces is a good example; here, one has $B_2 = 22$, $p_g = 1$, and thus $\rho \leq 20$. For instance, a K3 surface with $\rho \geq 12$ possesses an elliptic fibration with a section, and when $\rho \geq 17$ one can sometimes deduce the existence of an interesting algebraic correspondence (arising from a *Shioda–Inose structure*) between $X$ and an abelian surface [Mor1]. When $\rho \geq 19$, such a correspondence is guaranteed, and this has been used to deduce certain modularity results for $X$ when $k = \mathbb{Q}$ [Liv, Yui].

In this paper we seek examples of surfaces $X$ with $p_g = q = 1$ and $K^2 = 2$ with large Picard number. This interesting collection of surfaces of general type has been classified by Bombieri–Catanese [Cat] and Horikawa [Hor]. In particular, one knows that they are parametrized by an irreducible variety of dimension 7, with one of these dimensions accounting for the isomorphism class of the elliptic curve $\mathrm{Alb}(X)$. For such $X$, one has $2 \leq \rho \leq 10$, where the lower bound takes into consideration the classes of the canonical divisor and an Albanese fiber. In [LL], an explicit example is produced that shows this lower bound is sharp. In contrast, this note gives a one-parameter family of examples where $\rho$ is much closer to the upper bound:

2010 *Mathematics Subject Classification.* 14J29, 14C22, 14G25.

*Key words and phrases.* Surfaces of general type, Picard numbers.

**Theorem 1.1.** *Fix $\tau \in k$ such that*

$$\tau(\tau^2-1)(9\tau^2-1)(1+6\tau+\tau^2) \neq 0,$$

*and define the elliptic curve*

$$E_\tau: \quad y^2 = x^3 + (1-6\tau^2-3\tau^4)x^2 + 16\tau^6 x,$$

*as well as its quotient $\hat{E}_\tau := E_\tau/\langle(0,0)\rangle$. There exists an explicit surface $X_\tau$ with $p_g = q = 1, K^2 = 2$, and $\mathrm{Alb}(X_\tau) = \hat{E}_\tau$ whose Picard number over $k$ satisfies $\rho \geq 8$. Moreover, this bound cannot be improved in general, as $X_3$ has Picard number $\rho = 8$.*

Let us give some indication of the nature (specifically what we mean by "explicit") and proof of the theorem. By the aforementioned classification theorem, the Albanese map $X_\tau \to \hat{E}_\tau$ factors as $X_\tau \to \hat{\mathbf{B}} \to \hat{E}_\tau$, where $\hat{\mathbf{B}} \to E_\tau$ is a $\mathbb{P}^1$-bundle and $X_\tau \to \hat{\mathbf{B}}$ is a ramified double cover. Pulling these two maps back via $E_\tau \to \hat{E}_\tau$, one obtains a diagram

$$\begin{array}{ccc} Y_\tau & \longrightarrow & X_\tau \\ \downarrow & & \downarrow \\ \mathbf{B} & \longrightarrow & \hat{\mathbf{B}} \\ \downarrow & & \downarrow \\ E_\tau & \longrightarrow & \hat{E}_\tau, \end{array}$$

where each horizontal map is an unramified double cover. In particular, $X_\tau$ is the quotient of $Y_\tau$ by a certain free involution. We give explicit equations for the branch curve of the double cover $Y_\tau \to \mathbf{B}$ (see Theorem 5.3). The proof of the first statement in the theorem will follow from the fact that this branch curve possesses:

(1) Two $A_5$-singularities that map to a single $k$-rational $A_5$-singularity in the branch curve of $X_\tau \to \hat{\mathbf{B}}$. (This will imply the Picard number of $X_\tau$ is at least 7.)
(2) A certain tangency to another irreducible curve on $\mathbf{B}$, which will produce a reducible divisor in $Y_\tau$ that remains reducible when projected to $X_\tau$. (This will increase the lower bound from 7 to 8.)

The second statement in the theorem is proved using a method developed in [LL], namely by computing the zeta function of the reduction of $Y_3$ to $\mathbb{F}_{11}$, using this to infer the zeta function of $X_3$ over $\mathbb{F}_{11}$, and then deducing that 8 is also an upper bound for the Picard number of $X_3$.

By a result of Morrison [Mor2], to any surface $X$ with $p_g = q = 1$ and $K^2 = 2$ over $\mathbb{C}$, one may use Hodge theory to associate a K3 surface $Z_X$, unique up to isomorphism, whose transcendental part is isomorphic (as a rational Hodge structure) to that of $X$. (Here the *transcendental part* of a complex algebraic surface $Z$ refers to the orthogonal complement in $H^2(Z, \mathbb{Q})$ of the rational $(1,1)$-classes under the cup product form.) With $X_\tau$ as in Theorem 1.1, this means that $Z_{X_\tau}$ has Picard number at least 18. Given the aforementioned work of Morrison [Mor1] on K3 surfaces with large Picard number, it would be of interest to seek a nontrivial geometric correspondence between the surfaces $X_\tau$,a family of K3 surfaces, and (if possible) a family of abelian surfaces. In a more arithmetic direction, the surfaces $X_\tau$ are surfaces of general type that do not arise in any obvious way as Shimura varieties, and thus it would be interesting to show that $X_\tau$ is modular for at least some $\tau \in \mathbb{Q}$. The fact that their transcendental part has dimension at most 4 offers hope that current techniques from the theory of Galois representations might be able to achieve this.

Here is an outline of the paper. In §2, we describe how the classification of surfaces with $p_g = q = 1$ and $K^2 = 2$ allows one to view them as quotients of other surfaces by a free involution. Tractable equations for the latter class of surfaces are then developed in §3 and §4. In §5, we identify the surfaces $X_\tau$ from Theorem 1.1 (which are denoted as $X^-(\mu(\tau))$ in the body of the paper) and show that their Picard number is at least 7 (Corollary 5.4). We then increase this lower bound in §6 by identifying a special element in the Néron-Severi group (Proposition 6.1); it is worth noting this section also contains a small number of examples where $X_\tau$ has Picard number at least 9, but these are not defined over $\mathbb{Q}$ (Proposition 6.2). Finally, §7 shows that this lower bound is sharp for the surface $X_3$ (Corollary 7.2).

## 2. Quotient constructions of surfaces with $p_g = q = 1,\ K^2 = 2$

Let $k$ be a finitely-generated field of characteristic 0, and let $\bar{k}$ denote its algebraic closure. In independent work, Bombieri–Catanese and Horikawa classified algebraic surfaces over $\bar{k}$ with invariants $p_g = q = 1$ and $K^2 = 2$, by realizing them as certain branched double covers of symmetric squares of elliptic curves. Let $A$ be an elliptic curve over $\bar{k}$ and let $\mathrm{Sym}^2(A)$ denote its symmetric square. The Abel-Jacobi map

$$\mathrm{AJ} : \mathrm{Sym}^2(A) \to A, \qquad P + P' \mapsto P \oplus_A P',$$

gives $\mathrm{Sym}^2(A)$ the structure of a $\mathbb{P}^1$-bundle over $A$. Define on $\mathrm{Sym}^2(A)$ the invertible sheaf

$$\hat{\mathcal{L}} := \mathcal{O}_{\mathrm{Sym}^2(A)}(3) \otimes \mathrm{AJ}^* \mathcal{O}_A(-0_A).$$

**Theorem 2.1** ([Cat],[Hor])**.** *The following hold:*

(1) *The general element of the linear system $|\hat{\mathcal{L}}^{\otimes 2}|$ is a smooth curve on $\mathrm{Sym}^2(A)$, and $\Gamma(\mathrm{Sym}^2(A), \hat{\mathcal{L}}^{\otimes 2}) = 7$.*
(2) *If the curve $\Gamma \in |\hat{\mathcal{L}}^{\otimes 2}|$ has at most simple singularities, let $X'$ denote the minimal resolution of the double cover of $\mathrm{Sym}^2(A)$ inside the line bundle associated to $\hat{\mathcal{L}}$ that has branch locus $\Gamma$. Then $X'$ is the canonical model of a minimal algebraic surface $X$ with invariants $p_g = q = 1$ and $K^2 = 2$, and the composition $X \to X' \to \mathrm{Sym}^2(A) \to A$ is an Albanese fibration of $X$.*
(3) *Any minimal algebraic surface $Y$ over $\bar{k}$ with invariants $p_g = q = 1$, $K^2 = 2$, and $\mathrm{Alb}(Y) = A$ is isomorphic a surface $X$ arising from the construction above.*

*Remark* 1. In the statement of [LL, Theorem 2.1] (which is essentially the same as Theorem 2.1 above), the singularities of the branch curve should be corrected to read “simple singularities”, rather than “rational double points.”

Following [LL] (which draws from ideas in [Ish]), we now describe how one may use this classification to find explicit equations for unramified double covers of surfaces over $k$ with $p_g = q = 1$, $K^2 = 2$. Let $E$ be an elliptic curve over $k$ with a fixed $k$-rational 2-torsion point $P_1 \in E(k)$. We also let $P_0 := 0_E$ denote the identity element of $E$.

Let $\hat{E} := E/\langle P_1\rangle$ denote the quotient of $E$ by the translation $P \mapsto P \oplus_E P_1$, and let $\varphi : E \to \hat{E}$ denote the quotient map. Then $\hat{E}$ is an elliptic curve over $k$, and one may apply Theorem 2.1 to classify all surfaces over $\bar{k}$ with $p_g = q = 1$, $K^2 = 2$, and Albanese variety $\hat{E}_{\bar{k}}$. In doing so, one is led to consider the $\mathbb{P}^1$-bundle $\mathrm{Sym}^2(\hat{E}_{\bar{k}}) \to \hat{E}_{\bar{k}}$, and this $\mathbb{P}^1$-bundle turns out to be $\bar{k}$-isomorphic to the $\mathbb{P}^1$-bundle $\hat{\rho} : \hat{\mathbf{B}} \to \hat{E}$, where

$$\hat{\mathbf{B}} := \mathbf{Proj}\left(\mathrm{Sym}\left(\varphi_* \mathcal{O}_E(P_0)\right)\right).$$

By abuse of our earlier notation, we will write

$$\hat{\mathcal{L}} \quad := \quad \mathcal{O}_{\hat{\mathbf{B}}}(3) \otimes \hat{\rho}^* \mathcal{O}_{\hat{E}}(-0_{\hat{E}})$$

Hence classifying surfaces over $\bar{k}$ with $p_g = q = 1$, $K^2 = 2$, and Albanese variety $\hat{E}_{\bar{k}}$ comes down to studying the double covers obtained from the linear system $|\hat{\mathcal{L}}^{\otimes 2}|$ on $\hat{\mathbf{B}}_{\bar{k}}$.

Next we define the $\mathbb{P}^1$-bundle $\rho : \mathbf{B} \to E$ by

$$\mathbf{B} := \mathbf{Proj}\left(\mathrm{Sym}\big(\mathcal{O}_E(P_0) \oplus \mathcal{O}_E(P_1)\big)\right).$$

Given that

$$\mathcal{O}_E(P_0) \oplus \mathcal{O}_E(P_1) = \varphi^* \varphi_* \mathcal{O}_E(P_0), \tag{2.1}$$

we have a Cartesian square

$$\begin{tikzcd} \mathbf{B} \arrow[r, "\Phi"] \arrow[d, "\rho"'] & \hat{\mathbf{B}} \arrow[d, "\hat{\rho}"'] \\ E \arrow[r, "\varphi"] & \hat{E}. \end{tikzcd}$$

In particular, $\varphi$ and $\Phi$ are both unramified double covers. The translation $P \mapsto P \oplus_E P_1$ lifts to an involution $\iota : \mathbf{B} \to \mathbf{B}$, and the quotient of $\mathbf{B}$ by $\iota$ gives the map $\Phi$. Our primary reason for studying the unramified double cover $\mathbf{B}$ is that the sheaf in (2.1) easily yields local projective coordinates on $\mathbf{B}$; this is detailed in §3.

We will also define on $\mathbf{B}$ the invertible sheaf

$$\mathcal{L} := \Phi^* \hat{\mathcal{L}} = \mathcal{O}_{\mathbf{B}}(3) \otimes \rho^* \mathcal{O}_E(-P_0 - P_1). \tag{2.2}$$

We let

$$\hat{\pi} : \hat{L} = \mathbf{Spec}(\mathrm{Sym}(\hat{\mathcal{L}}^\vee)) \to \hat{\mathbf{B}}, \qquad \pi : L = \mathbf{Spec}(\mathrm{Sym}(\mathcal{L}^\vee)) \to \mathbf{B}$$

denote the projection from the total spaces of $\hat{\mathcal{L}}$ and $\mathcal{L}$, respectively. As $L$ is the pullback of $\hat{L}$ via $\Phi$, we have an unramified double covering $\Psi : L \to \hat{L}$ that fits into the diagram

$$\begin{tikzcd} L \arrow[r, "\Psi"] \arrow[d, "\pi"'] & \hat{L} \arrow[d, "\hat{\pi}"] \\ \mathbf{B} \arrow[r, "\Phi"] \arrow[d, "\rho"'] & \hat{\mathbf{B}} \arrow[d, "\hat{\rho}"] \\ E \arrow[r, "\varphi"] & \hat{E}. \end{tikzcd}$$

We let $\iota^+ : L \to L$ denote the pullback of the free involution $\iota : \mathbf{B} \to \mathbf{B}$; thus the quotient of $L$ by $\iota^+$ gives $\Psi$. Since $\Psi = \Psi \circ \iota^+$, we see the invertible sheaf $\pi^*\mathcal{L}$ on $L$ is preserved by $(\iota^+)^*$. Let us define a second free involution $\iota^-$ by $\iota^- = \iota^+ \circ [-1] = [-1] \circ \iota^+$, where $[-1] : L \to L$ denotes the automorphism giving multiplication by $-1$ on each fiber; then $(\iota^-)^*$ also preserves $\pi^*\mathcal{L}$.

Later we shall need:

**Lemma 2.2.** *Let $W \in \Gamma(L, \pi^*\mathcal{L})$ denote the tautological section of $\pi^*\mathcal{L}$. Then $(\iota^+)^*(W) = W$ and $(\iota^-)^*(W) = -W$.*

*Proof.* The $\mathbf{B}$-scheme $L$ represents the functor from $\mathbf{B}$-schemes to abelian groups given by

$$(f : T \to \mathbf{B}) \quad \mapsto \quad \Gamma(T, f^*\mathcal{L}),$$

see [GW, Prop. 11.3], and the tautological section $W$ corresponds to $\mathrm{Id}_L \in \mathrm{Hom}_{\mathbf{B}}(L, L)$. A similar statement applies to the tautological section $\hat{W}$ of $\hat{\pi}^*\hat{L}$, and hence we have $\Psi^*\hat{W} = W$. Given that $\Psi$ is the quotient map of $L$ by $\iota^+$, we conclude $(\iota^+)^*$ fixes $W$. On the other hand, $[-1]^*W = -W$, and so $(\iota^-)^*W = -W$ as well. □

Since we have an isomorphism

$$\Phi^* : \ \Gamma(\hat{\mathbf{B}}, \hat{\mathcal{L}}^{\otimes 2}) \xrightarrow{\sim} \left[\Gamma(\mathbf{B}, \mathcal{L}^{\otimes 2})\right]^{\iota},$$

we will set $\hat{s} := (\Phi^*)^{-1}(s)$ whenever $s \in \left[\Gamma(\mathbf{B}, \mathcal{L}^{\otimes 2})\right]^{\iota}$; we note that, as $\hat{\mathcal{L}}$, $\mathcal{L}$, and $\Phi$ are defined over $k$, the isomorphism above is one of $k$-vector spaces (of dimension 7, by Theorem 2.1). By taking divisors of zeros, the sections also $s$ and $\hat{s}$ yield effective divisors

$$Z(s) \in |\mathcal{L}^{\otimes 2}|, \qquad Z(\hat{s}) \in |\hat{\mathcal{L}}^{\otimes 2}|.$$

**Definition 2.3.** *Given a section $s \in \left[\Gamma(\mathbf{B}, \mathcal{L}^{\otimes 2})\right]^{\iota}$, we let $Y_0(s) \to \mathbf{B}$ denote the double cover of $\mathbf{B}$ inside $L$ ramified over $Z(s)$; more precisely, $Y_0(s)$ is the divisor of zeros of the section*

$$W^2 - \pi^*s \in \Gamma(L, \pi^*\mathcal{L}^{\otimes 2}).$$

*Similarly, let $X_0^+(s) \to \hat{\mathbf{B}}$ be the double cover of $\hat{\mathbf{B}}$ inside $\hat{L}$ ramified over $Z(\hat{s})$.*

*Remark* 2. When working over a non-closed field $k$, the definition of $Y_0(s)$ depends not only upon the branch curve $Z(s)$, but also upon the section $s$. Indeed, if $a \in k^\times \setminus (k^\times)^2$ is a nonsquare element, then the section $W^2 - \pi^*(as)$ gives a surface $Y_0(as)$ that need not be isomorphic to $Y_0(s)$ over $k$.

If, as in the proof of Lemma 2.2, we let $\hat{W} \in \Gamma(\hat{L}, \hat{\pi}^*\hat{\mathcal{L}})$ denote the tautological section, then the defining sections of $Y_0(s)$ and $X_0^+(s)$ are related by

$$W^2 - \pi^* s = \Psi^*(\hat{W}^2 - \hat{\pi}^*\hat{s}).$$

Hence we have a commutative diagram

$$\begin{tikzcd} Y_0(s) \arrow[r, "q^+"] \arrow[d] & X_0^+(s) \arrow[d] \\ \mathbf{B} \arrow[r, "\Phi"] \arrow[d] & \hat{\mathbf{B}} \arrow[d] \\ E \arrow[r, "\varphi"] & \hat{E} \end{tikzcd} \tag{2.3}$$

in which the top square is Cartesian. Thus $q^+$ is an unramified double covering map; in fact, $q^+$ is the quotient map of $Y_0(s)$ by the restriction of $\iota^+ \in \mathrm{Aut}(L)$ to $Y_0(s)$; by abuse of notation, we again denote this restriction by $\iota^+$. Similarly, we still use $\iota^-$ to denote the restriction of $\iota^- \in \mathrm{Aut}(L)$ to $Y_0(s)$. Note that $h = \iota^+ \circ \iota^- = \iota^- \circ \iota^+$ is the hyperelliptic involution on $Y_0(s)$ over $\mathbf{B}$, and thus the subgroup $\{\mathrm{Id}, h, \iota^+, \iota^-\} \subseteq \mathrm{Aut}(Y_0(s))$ is isomorphic to the Klein 4-group.

If $S \to T$ is a double covering of surfaces over $k$, with $T$ nonsingular, and (reduced) branch locus $\Gamma \subseteq T$, one may form its *canonical resolution* (see [BHPVdV, III.7]), which is a double cover $\tilde{S} \to \tilde{T}$ forming a Cartesian square

$$\begin{tikzcd} \tilde{S} \arrow[r] \arrow[d] & S \arrow[d] \\ \tilde{T} \arrow[r] & T \end{tikzcd} .$$

Briefly, $\tilde{S} \to \tilde{T}$ is obtained by applying the $\sigma$-process simultaneously at all points of the singular locus of $\Gamma \subseteq T$ to obtain $T_1 \to T$, letting $S_1$ be the normalization of $S \times_T T_1$ so that $S_1 \to T_1$ is a new double cover with its own branch locus $\Gamma_1$, and repeating this process as necessary until the covering surface is nonsingular. When the branch locus $\Gamma$ of $S \to T$ has at most simple singularities, the resolution $\tilde{S} \to S$ is minimal, in the sense that no $(-1)$-curves are contracted via this map.

**Definition 2.4.** *For $s \in \left[\Gamma(\mathbf{B}, \mathcal{L}^{\otimes 2})\right]^{\iota}$, let $Y(s)$ denote the canonical resolution of $Y_0(s)$ obtained from the double covering $Y_0(s) \to \mathbf{B}$.*

The involutions $\iota^+, \iota^-, h$ on $Y_0(s)$ extend uniquely (via pullback by the resolution map $Y(s) \to Y_0(s)$) to involutions on $Y(s)$, and we use the same notation to denote these elements of $\mathrm{Aut}(Y(s))$. Note that $\iota^+$ and $\iota^-$ are both free.

**Definition 2.5.** *For $s \in \left[\Gamma(\mathbf{B}, \mathcal{L}^{\otimes 2})\right]^{\iota}$, we define the two quotient surfaces*

$$X^+(s) := Y(s)/\iota^+, \qquad X^-(s) := Y(s)/\iota^-.$$

*Remark* 3. For statements that apply to both $X^+(s)$ and $X^-(s)$, or to $\iota^+$ and $\iota^-$, we will frequently use the notation $X^\varepsilon(s)$ and $\iota^\varepsilon$, with the understanding that $\varepsilon \in \{+, -\}$.

Since the composition $Y(s) \to Y_0(s) \to \mathbf{B} \to E \to \hat{E}$ is invariant under $\iota^\varepsilon$, we have a fibration $X^\varepsilon(s) \to \hat{E}$.

**Theorem 2.6.** *Assume that $s \in \left[\Gamma(\mathbf{B}, \mathcal{L}^{\otimes 2})\right]^\iota$ has the following two properties:*

- *The divisor $Z(s)$ on $\mathbf{B}$ has at most simple singularities, all defined over $k$.*
- *The fibration $Y(s) \to E$ is relatively minimal (or, equivalently, the surface $Y(s)$ is minimal).*

*Then the surface $X^\varepsilon(s)$ is a smooth minimal surface over $k$ with invariants $p_g = q = 1$, $K^2 = 2$, and Albanese variety $\hat{E}$.*

*Proof.* Given that the divisor $Z(s)$ in $\mathbf{B}$ is an unramified double cover of the divisor $Z(\hat{s})$ in $\hat{\mathbf{B}}$, both have at most simple singularities. From Theorem 2.1 it follows that $X_0^+(s)$ (which was defined as the double cover of $\hat{\mathbf{B}}$ inside $\hat{L}$ ramified over $Z(\hat{s})$) is the canonical model of a surface over $k$ with $p_g = q = 1, K^2 = 2$.

We will first show the that, up to isomorphism over $k$, the surface $X^+(s)$ is the canonical resolution of $X_0^+(s)$ obtained from the double cover $X_0^+(s) \to \hat{\mathbf{B}}$. Let $X_1 \to \hat{\mathbf{B}}_1$ denote the canonical resolution obtained from $X_0^+(s) \to \hat{\mathbf{B}}$ (as described before Definition 2.4). Thus we have a commutative diagram

$$\begin{tikzcd}
X_1 \arrow[r] \arrow[d] & X_0^+(s) \arrow[d] \\
\hat{\mathbf{B}}_1 \arrow[r] & \hat{\mathbf{B}}
\end{tikzcd}$$

where the map $\hat{\mathbf{B}}_1 \to \hat{\mathbf{B}}$ is a certain sequence of $\sigma$-processes; furthermore, since $Z(\hat{s})$ has only simple singularities (and thus no normalization is required in forming the canonical resolution), this diagram is Cartesian. But as the branch locus of $Y_0(s) \to \mathbf{B}$ is $Z(s) = Z(\Phi^*\hat{s}) = \Phi^* Z(\hat{s})$, it follows that the base change of this sequence of $\sigma$-processes $\mathbf{B} \to \hat{\mathbf{B}}$ will also yield the canonical resolution of $Y_0(s) \to \mathbf{B}$. That is, if we define $\mathbf{B}_1 = \mathbf{B} \times_{\hat{\mathbf{B}}} \hat{\mathbf{B}}_1$, then we have cubical diagram

(2.4)
$$\begin{tikzcd}
& Y(s) \arrow[rr] \arrow[dl] \arrow[dd] & & Y_0(s) \arrow[dl, "q^+"] \arrow[dd] \\
X_1 \arrow[rr] \arrow[dd] & & X_0^+(s) \arrow[dd] & \\
& \mathbf{B}_1 \arrow[rr] \arrow[dl] & & \mathbf{B} \arrow[dl, "\Phi"] \\
\hat{\mathbf{B}}_1 \arrow[rr] & & \hat{\mathbf{B}} &
\end{tikzcd}$$

where all faces of this cube commute and the bottom, front, back, and right faces are Cartesian. hence a diagram chase shows that top face is Cartesian as well. Therefore, by contracting the top and right faces of the cube, we arrive at the Cartesian diagram

$$\begin{tikzcd} Y(s) \arrow[r] \arrow[d] & X_1 \arrow[d] \\ \mathbf{B} \arrow[r, "\Phi"] & \hat{\mathbf{B}}, \end{tikzcd}$$

showing that the covering $Y(s) \to X_1$ is the pullback of $\Phi : \mathbf{B} \to \hat{\mathbf{B}}$, which is the quotient map of $\mathbf{B}$ by $\iota$; hence $Y(s) \to X_1$ is the quotient map of $Y(s)$ by $\iota^+$, and so $X^+(s)$ is isomorphic to the canonical resolution of $X_0^+(s)$. Finally, this isomorphism is defined over $k$, as the same is true of all of the maps considered in this paragraph.

From the diagram (2.3) and our discussion of the cube (2.4), it follows that

$$\begin{tikzcd} Y(s) \arrow[r] \arrow[d] & X^+(s) \arrow[d] \\ E \arrow[r, "\varphi"] & \hat{E} \end{tikzcd} \tag{2.5}$$

is Cartesian. Thus the hypothesis that $Y(s) \to E$ is relatively minimal implies the same for $X^+(s) \to \hat{E}$, and therefore $X^+(s)$ is minimal. By the remark from the beginning of the proof, the theorem holds for $X^+(s)$.

It remains to verify the theorem for $X^-(s)$. As $Y(s) \to X^+(s)$ is an unramified double cover, we have

$$\begin{aligned} K^2_{Y(s)} &= 2K^2_{X^+(s)} = 4, \\ \chi(\mathcal{O}_{Y(s)}) &= 2\chi(\mathcal{O}_{X^+(s)}) = 2, \\ e(Y(s)) &= 2e(X^+(s)) = 20. \end{aligned}$$

Moreover, one can use (2.5) to show (see [LL, Lemma 3.3]) that $\mathrm{Alb}(Y(s)) = E$, and thus $Y(s)$ has irregularity $q = 1$. Since $\iota^-$ is a free involution, it follows that the quotient map $Y(s) \to X^-(s)$ is also an unramified double cover. Hence the invariants $K^2, \chi, e$ of $X^-(s)$ will match those of $X^+(s)$. Moreover, the diagram

$$\begin{tikzcd} Y(s) \arrow[r] \arrow[d] & X^-(s) \arrow[d] \\ \mathbf{B} \arrow[r, "\Phi"] \arrow[d, "\rho"'] & \hat{\mathbf{B}} \arrow[d, "\hat{\rho}"'] \\ E \arrow[r, "\varphi"] & \hat{E} \end{tikzcd}$$

and the fact that $\mathrm{Alb}(Y(s)) = E$ allow us to conclude that $\mathrm{Alb}(X^-(s)) = \hat{E}$. This shows that $X^-(s)$ satisfies $p_g = q = 1$, $K^2 = 2$, and finally the minimality of $Y(s)$ implies that of $X^-(s)$. $\square$

*Remark* 4. The surfaces $X^+(s)$ and $X^-(s)$ are both double covers of $\hat{\mathbf{B}}$ with the same branch locus, but in general they are not isomorphic over $\bar{k}$. This indicates that they live inside different line bundles over $\hat{E}$. While $X^+(s)$ lives inside the total space of $\hat{\mathcal{L}}$, one may show (using that fact that both surfaces are quotients of $Y(s)$) that $X^-(s)$ lives inside the total space of $\hat{\mathcal{L}} \otimes (\hat{\rho})^*(\mathcal{O}_{\hat{E}}(0_{\hat{E}} - Q))$, where $\varphi(E[2]) = \{0_{\hat{E}}, Q\}$.

## 3. Coordinates for $\mathbf{B}$ and $L$

The reason for studying the unramified double cover $\mathbf{B}$ is that the locally free sheaf in (2.1) decomposes into two fairly simple invertible sheaves on $E$, and this yields convenient local coordinates on $\mathbf{B}$. Given that

$$\Gamma(\mathbf{B}, \mathcal{O}_{\mathbf{B}}(1)) = \Gamma(E, \mathcal{O}_E(P_0)) \oplus \Gamma(E, \mathcal{O}_E(P_1)), \tag{3.1}$$

we let $Z_0 \in \Gamma(\mathbf{B}, \mathcal{O}_{\mathbf{B}}(1))$ (resp. $Z_1 \in \Gamma(\mathbf{B}, \mathcal{O}_{\mathbf{B}}(1))$) correspond to the element $(1, 0)$ (resp. $(0, 1)$) on the right side; note that in fact

$$Z_i \in \Gamma(\mathbf{B}, \mathcal{O}_{\mathbf{B}}(1) \otimes \rho^*\mathcal{O}_E(-P_i)) \tag{3.2}$$

for $i = 0, 1$. Since

$$\left[\Gamma(\mathbf{B}, \mathcal{O}_{\mathbf{B}}(1))\right]^{\iota} = \Gamma(\hat{\mathbf{B}}, \mathcal{O}_{\hat{\mathbf{B}}}(1)) = \Gamma(\hat{E}, \varphi_*\mathcal{O}_E(P_0)) = \Gamma(E, \mathcal{O}_E(P_0)) \simeq k,$$

it follows that $Z_0 + Z_1$ is the pullback of the unique (up to constant multiple) nonzero section in $\Gamma(\hat{\mathbf{B}}, \mathcal{O}_{\hat{\mathbf{B}}}(1))$. In particular, the divisor $Z(Z_0 + Z_1)$ is the unique $\iota$-invariant tautological divisor in $\mathbf{B}$.

If we define on $E$ the open subset $U := E \setminus \{P_0, P_1\}$, then $\mathbf{B}$ is trivial on $\mathbf{B}_{|U} = \pi^{-1}(U)$, and $(Z_0 : Z_1)$ gives relative projective coordinates there. For this reason, we will write points on $\mathbf{B}|_U$ in the form $(P, (Z_0 : Z_1))$, where $P \in U$ and $(Z_0 : Z_1) \in \mathbb{P}^1$. The fibers $\mathbf{B}_{P_0}$ and $\mathbf{B}_{P_1}$ are exchanged by $\iota$, and hence $\iota$ preserves $\mathbf{B}_{|U}$; on this subset, it takes the form

$$\iota(P, (Z_0 : Z_1)) = (P \oplus_E P_1, (Z_1 : Z_0)).$$

To obtain a complete open cover of $\mathbf{B}$, we also need open sets that cover $\mathbf{B}_{P_0}$ and $\mathbf{B}_{P_1}$. For the objects we will work with, it will suffice to work with an infinitesimal neighborhood of $\mathbf{B}_{P_0}$, and then appeal to $\iota$-invariance in order to infer what happens in a neighborhood of $\mathbf{B}_{P_1}$. With this in mind, let $t \in k(E)$ denote a local parameter at $P_0$ on $E$ (which we make explicit in (4.2) below). Recalling from (3.2) that $Z_0$ vanishes on $\mathbf{B}_{P_0}$, if $U'$ is a sufficiently small neighborhood of $P_0$ and we put $Z_0' := t^{-1}Z_0$, then $(Z_0' : Z_1)$ gives relative projective coordinates on $\mathbf{B}_{|U'}$. Since $t$ is a local parameter at $P_0$, we can write points of $\mathbf{B}$ near $\mathbf{B}_{P_0}$ in the form in the form $(t, (Z_0' : Z_1))$, with $(Z_0' : Z_1) \in \mathbb{P}^1$.

## 4. Equations for $Z(s)$ and $Y_0(s)$

The surfaces $Y(s), X^+(s)$, and $X^-(s)$ defined in §2 arise from $\iota^*$-invariant sections of the invertible sheaf $\mathcal{L}$ in (2.2). We now recall a convenient description of these sections from [LL, Prop. 3.1]. First choose a Weierstrass equation for $E$ over $k$:

$$E\colon\quad y^2 = c(x) = (x-\alpha_1)(x-\alpha_2)(x-\alpha_3),$$

where $c(x) \in k[x]$ is a nonsingular cubic. We let $P_i = (\alpha_i, 0)$, $i = 1, 2, 3$, denote the nontrivial 2-torsion points of $E$. If $a := c'(\alpha_1)$, $b := \frac{1}{2}c''(\alpha_1)$ then we may also write $E$ in the form

$$E : y^2 = a(x-\alpha_1) + b(x-\alpha_1)^2 + (x-\alpha_1)^3. \tag{4.1}$$

Define on $E$ the following four rational functions:

$$\begin{aligned} g_0(P) &= x(P) - x(P_1) \\ &= x - \alpha_1 \\ h_0(P) &= y(P) \\ &= y \\ g_1(P) &= g_0(P \oplus_E P_1) \\ &= \frac{a}{x-\alpha_1} \\ h_1(P) &= h_0(P \oplus_E P_1) \\ &= -\frac{ay}{(x-\alpha_1)^2} \end{aligned}$$

By considering the divisors of $g_0, h_0, g_1, h_1$ and also using (3.1), one may show:

**Proposition 4.1.** *A basis of* $\left[\Gamma(\mathbf{B}, \mathcal{L}^{\otimes 2})\right]^{\iota}$ *is given by the following seven sections:*

$$\begin{aligned} \psi_0 &:= g_0 Z_0^6 + g_1 Z_1^6 \\ \psi_1 &:= g_0^2 Z_0^6 + g_1^2 Z_1^6 \\ \psi_2 &:= g_0 Z_0^5 Z_1 + g_1 Z_0 Z_1^5 \\ \psi_3 &:= h_0 Z_0^5 Z_1 + h_1 Z_0 Z_1^5 \\ \psi_4 &:= Z_0^4 Z_1^2 + Z_0^2 Z_1^4 \\ \psi_5 &:= g_0 Z_0^4 Z_1^2 + g_1 Z_0^2 Z_1^4 \\ \psi_6 &:= Z_0^3 Z_1^3. \end{aligned}$$

If $s \in \left[\Gamma(\mathbf{B}, \mathcal{L}^{\otimes 2})\right]^{\iota}$, the expressions for the sections $\psi_i$ given here work well for analyzing the divisor $Z(s)$ on $\mathbf{B}_{|U}$.

We now develop equations for analyzing $Z(s)$ in a neighborhood of the fiber $\mathbf{B}_{P_0}$. Define

$$t := \frac{x-\alpha_1}{y} = \frac{g_0}{h_0} = -\frac{g_1}{h_1}, \tag{4.2}$$

which is a local parameter at $P_0$ on $E$. As discussed in §3, we will write points on $\mathbf{B}$ near $\mathbf{B}_{P_0}$ as $(t, (Z_0' : Z_1))$. Expanding the functions $g_i, h_i$ in terms of $t$, one finds:

$$\begin{aligned} g_0 &= \frac{1}{t^2} - b - at^2 + O(t^3) \\ h_0 &= \frac{1}{t} g_0 \\ g_1 &= at^2 + abt^4 + O(t^6) \\ h_1 &= -\frac{1}{t} g_1 \end{aligned}$$

If we write $s$ as a linear combination of the sections $\psi_i$, we may obtain an equation for $Z(s)$ near $\mathbf{B}_{P_0}$ concretely by substituting the expansions above for $g_0, h_0, g_1, h_1$ and the equality $Z_0 = tZ_0'$ into the equations in Proposition 4.1, and then dividing by $t^2$. Hence if $s = \sum_i a_i \psi_i$, then $Z(s)$ is given in a neighborhood of $\mathbf{B}_{P_0}$ by

$$\sum a_i \chi_i = 0,$$

where

$$\begin{aligned} \chi_0 &= aZ_1^6 + t^2\big((Z_0')^6 + abZ_1^6\big) + O(t^4) \\ \chi_1 &= (Z_0')^6 + t^2(-2b(Z_0')^6 + a^2 Z_1^6) + O(t^4) \\ \chi_2 &= t((Z_0')^5 Z_1 + aZ_0' Z_1^5) + t^3(-b(Z_0')^5 Z_1 + abZ_0' Z_1^5) + O(t^5) \\ \chi_3 &= (Z_0')^5 Z_1 - aZ_0' Z_1^5 + t^2(-b(Z_0')^5 Z_1 - abZ_0' Z_1^5) + O(t^4) \\ \chi_4 &= (Z_0')^2 Z_1^4 + t^2 (Z_0')^4 Z_1^2 \\ \chi_5 &= (Z_0')^4 Z_1^2 + t^2(-b(Z_0')^4 Z_1^2 + a(Z_0')^2 Z_1^4) + O(t^4) \\ \chi_6 &= t(Z_0')^3 Z_1^3 \end{aligned}$$

Finally, we note that since $\iota^* s = s$ and $\iota$ exchanges $\mathbf{B}_{P_0}$ and $\mathbf{B}_{P_1}$, the behavior of $Z(s)$ near $\mathbf{B}_{P_1}$ can by inferred by our analysis near $\mathbf{B}_{P_0}$. Hence the polynomials $\psi_i \in k(U)[Z_0, Z_1]$ and $\chi_i \in k((t))[Z_0', Z_1]$ will give us a complete picture of $Z(s)$.

To finish this section, we discuss equations for the surface $Y_0(s)$, with $s = \sum_{i=0}^6 a_i \psi_i$. The restriction of $Y_0(s)$ to $\mathbf{B}|_U \cap \{Z_1 \neq 0\}$ is given (in $L$) by an equation of the form

$$T_0^2 = \sum_{i=0}^6 a_i \psi_i\Big(\frac{Z_0}{Z_1}, 1\Big),$$

and its restriction to $\mathbf{B}|_U \cap \{Z_0 \neq 0\}$ is given by

$$T_1^2 = \sum_{i=0}^6 a_i \psi_i\Big(1, \frac{Z_1}{Z_0}\Big).$$

But it is simpler to unify these two local equations by considering the restriction of $Y_0(s)$ to $\mathbf{B}|_U$ as a family of curves in weighted projective space: it may be identified with the family

$$T^2 = \sum_{i=0}^6 a_i \psi_i(Z_0, Z_1) \tag{4.3}$$

in $U \times \mathbb{P}^2(1,1,3)$. By Lemma 2.2, in these coordinates the involutions $\iota^+, \iota^-, h \in \mathrm{Aut}(Y_0(s))$ take the form

$$\begin{aligned} \iota^+(P,(Z_0:Z_1:T)) &= (P \oplus_E P_1,(Z_1:Z_0:T)) \\ \iota^-(P,(Z_0:Z_1:T)) &= (P \oplus_E P_1,(Z_1:Z_0:-T)) \\ h(P,(Z_0:Z_1:T)) &= (P,(Z_0:Z_1:-T)) \end{aligned}$$

In a neighborhood of $\pi^{-1}(\mathbf{B}_{P_0})$, $Y_0(s)$ is given by the equation

$$(T')^2 = \sum_{i=0}^{6} a_i \chi(t',(Z_0:Z_1)). \tag{4.4}$$

In particular, the fiber $Y_0(s)_{P_0} \subseteq \mathbb{P}^2(1,1,3)$ is obtained by setting $t = 0$ in (4.4). Moreover, the fiber $Y_0(s)_{P_1}$ is isomorphic to $Y_0(s)_{P_0}$ over $k$.

## 5. An interesting linear subsystem

In this section we will study a linear subsystem of $|\mathcal{L}^{\otimes 2}|$ whose generic elements (i) are all singular at a special point of $\mathbf{B}$ and (ii) yield surfaces $Y(s)$ whose Néron-Severi group is not generated solely by the canonical divisor, an Albanese fiber, and exceptional curves. By searching for elements where the singularity is more extreme, we can find examples of surfaces with $p_g = q = 1, K^2 = 2$ that have large Picard number.

First, for $\tau \in k$, define

$$E_\tau : \quad y^2 = x^3 + (1 - 6\tau^2 - 3\tau^4)x^2 + 16\tau^6 x.$$

The right side has discriminant $\Delta(\tau) = 256\tau^{12}(\tau^2-1)^3(9\tau^2-1)$. Letting

$$S = \{0, \pm 1, \pm 1/3\}, \tag{5.1}$$

we will assume in what follows that $\tau \notin S$, i.e., that $E_\tau$ is an elliptic curve. We let $P_1 := (\alpha_1, 0) = (0,0)$ so that, in the notation (4.1), we have

$$a = -16\tau^6, \qquad b = 1 - 6\tau^2 - 3\tau^4.$$

We also note that, for a general point $(x,y) \in E_\tau$, translation by $P_1$ takes the form

$$(x,y) \oplus_{E_\tau} P_1 = (g_1, h_1) = \left( \frac{16\tau^6}{x}, -\frac{16\tau^6 y}{x^2} \right)$$

The curve $E_\tau$ is also equipped with a 6-torsion point

$$Q := (x,y) = (4\tau^2, 4\tau^2(\tau^2-1)),$$

which satisfies $3Q = P_1$.

Up to scalar multiplication, $Z_0 + Z_1$ is the unique element of $\left[\Gamma(\mathbf{B}, \mathcal{O}_{\mathbf{B}}(H))\right]^\iota$. Hence $H := Z(Z_0 + Z_1)$ is the pullback of the unique tautological divisor on $\hat{\mathbf{B}}$. We let $\Lambda \subseteq \Gamma(\mathbf{B}, \mathcal{L}^{\otimes 2})^\iota$ denote the subspace of sections whose divisors are tangent to $H$ at the two points $(Q, (-1:1))$ and $(-Q, (-1:1))$ in $\mathbf{B}|_U$. If we put $Z := Z_0/Z_1$ then at both of these points local parameters are given by $(x - 4\tau^2, Z + 1)$. Note that the local equation of $H$ near these points is $Z + 1 = 0$, so more precisely we will say $\psi \in \Lambda$ exactly when

$$\psi(Q, (-1:1)) = \frac{\partial \psi}{\partial x}(Q, (-1:1)) = \psi(-Q, (-1:1)) = \frac{\partial \psi}{\partial x}(-Q, (-1:1)) = 0.$$

(Note that as $y$ may appear in $\psi$, $\partial\psi/\partial x$ is computed by implicit differentiation, with $\partial y/\partial x$ being determined by the equation of $E_\tau$.) Since each $\psi$ is a linear combination of the basis elements $\psi_0, \dots, \psi_6$, it follows that $\lambda$ is the kernel of the 4-by-7 matrix

$$\begin{bmatrix} \psi_j(Q, (-1:1)) \\ \psi_j(-Q, (-1:1)) \\ \frac{\partial \psi_j}{\partial x}(Q, (-1:1)) \\ \frac{\partial \psi_j}{\partial x}(-Q, (-1:1)) \end{bmatrix}_{0 \le j \le 6}$$

One may calculate this kernel to conclude that (for any value of $\tau \notin S$) the following four sections give a basis for $\Lambda$:

$$\begin{aligned}
\lambda_0 &:= \psi_0 + \psi_2 \\
&= (Z_0 + Z_1)(g_0 Z_0^5 + g_1 Z_1^5) \\
\lambda_1 &:= \psi_0 - \psi_5 \\
&= (Z_0 + Z_1)(Z_0 - Z_1)(g_0 Z_0^4 - g_1 Z_1^4) \\
\lambda_2 &:= \psi_4 + 2\psi_6 \\
&= (Z_0 + Z_1)^2 Z_0^2 Z_1^2 \\
\lambda_3 &:= -8\tau^2(\tau^2 + 1)\psi_0 + \psi_1 + 4\tau^4(1 + 4\tau^2 + \tau^4)(\psi_4 - 2\psi_6).
\end{aligned}$$

Note that $\lambda_0, \lambda_1, \lambda_2$ are in the image of the inclusion

$$\left[\Gamma(\mathbf{B}, \mathcal{L}^{\otimes 2} \otimes \mathcal{O}_{\mathbf{B}}(-H))\right]^\iota \hookrightarrow \left[\Gamma(\mathbf{B}, \mathcal{L}^{\otimes 2})\right]^\iota,$$

and so the "tangency" of $Z(\lambda_i)$ (for $i = 0, 1, 2$) to $H$ at $(\pm Q, (-1:1))$ is actually due to the fact that $H \subseteq \mathrm{Supp}(Z(\lambda_i))$. But the same is not true of $Z(\lambda_3)$, and so the tangency condition is a "genuine" one for generic $\lambda \in \Lambda$. This tangency condition will play a role later in §6, where we consider its effect on the Picard numbers of the surfaces $Y(\lambda)$ and $X^\varepsilon(\lambda)$, for $\lambda \in \Lambda$. For now, we focus on singularities of the divisors $Z(\lambda)$.

Observe that $\Lambda \subseteq \mathrm{Span}\big(\{\psi_0, \dots, \psi_6\} \setminus \{\psi_3\}\big)$, which implies (since only $\psi_3$ involves the functions $h_0, h_1$) that any $\lambda \in \Lambda$ satisfies the additional symmetry

$$\lambda(P, (Z_0 : Z_1)) = \lambda(-P, (Z_0 : Z_1)).$$

Hence, for any $P \in E$, the local behavior of $Z(\lambda)$ will be the same at each of the four points $(\pm P, (-1:1)), (\pm P \oplus_{E_\tau} P_1, (-1:1))$. In particular, $Z(\lambda)$ will be tangent to $H$ at the four points $(\pm Q \oplus_{E_\tau} P_1, (-1:1)) = (\pm 2Q, (-1:1))$. Since $H.Z(\lambda) = 8$, it follows, for generic $\lambda$, that the divisor $Z(\lambda)$ intersects $H$ only at the four points

$$\{(\pm Q, (-1:1)), (\pm 2Q, (-1:1))\} \subseteq \mathbf{B}|_U, \tag{5.2}$$

and that the local intersection number at each of these points is 2.

**Proposition 5.1.** *The base points of the linear system $|\Lambda|$ are precisely the four points in (5.2).*

*Proof.* One may start by checking that $Z(\lambda_0) \cap Z(\lambda_2) \cap \mathbf{B}_U = H \cap \mathbf{B}_U$. Since a generic $Z(\lambda)$ intersects $H$ only in the four points (5.2), this shows there are no other base points in the open set $\mathbf{B}|_U$.

Next one may check that $Z(\lambda_0) \cap Z(\lambda_1) \cap Z(\lambda_2) \cap \mathbf{B}_{P_0}$ consists of the single point $(t, (Z_0' : Z_1)) = (0, (1 : 0))$. But this is a point on $H = Z(tZ_0' + Z_1)$. Since a generic $Z(\lambda)$ does not intersect $H$ in the fiber $\mathbf{B}_{P_0}$, $|\Lambda|$ has no base points in $\mathbf{B}_{P_0}$. By $\iota$-invariance, the same is true in the fiber $\mathbf{B}_{P_1}$. □

**Corollary 5.2.** *A general element of $|\Lambda|$ is smooth.*

*Proof.* Bertini's theorem shows this is true away from the four base points. On the other hand, for a given value of $\tau$ one may find $\lambda$ such that $Z(\lambda)$ is smooth at each of these base points:

- When $4\tau^2 \neq \pm 1$ and $4\tau^4 \neq \pm 1$, one may take $\lambda = \lambda_0$, since $Z(g_0 Z_0^5 + g_1 Z_1^5)$ does not meet $H$ at any of the base points points.
- When $\tau^2 \neq \pm 1$, one may take $\lambda = \lambda_3$.

□

Hence the set of singular elements in $|\Lambda|$ has codimension at least one. If $Z(\lambda)$ has a singularity at a general point $(P, (Z_0 : Z_1)) \in \mathbf{B}|_U$, then it is in fact singular at the four points

$$(\pm P, (Z_0 : Z_1)), (\pm P \oplus_{E_\tau} P_1, (Z_1 : Z_0)). \tag{5.3}$$

However, if we fix $\sigma \in \bar{k}$ such that $\sigma^2 = (1-\tau)(1+3\tau)$, then the point

$$R := (4\tau^3, 4\tau^3(\tau - 1)\sigma) \in E_\tau(k(\sigma)) \tag{5.4}$$

satisfies $R \oplus_{E_\tau} P_1 = -R$; hence if $Z(\lambda)$ is singular at $(R, (1 : 1))$, then this only implies a singularity at the second point $(-R, (1 : 1))$. Heuristically, due to this "collapse" of four singularities into two, one therefore hopes that those $Z(\lambda)$ which are singular at $(R, (1 : 1))$ might offer a richer set of singularities to exploit.[1] This turns out to be fruitful:

**Theorem 5.3.** *If $\tau \notin S$ and $1 + 6\tau + \tau^2 \neq 0$, define $\mu(\tau) := \sum_{i=0}^{3} c_0 \lambda_0 \in \Lambda$, where*

$$\begin{aligned}
c_0 &= 4(1-\tau)^2\tau^2(1 + 14\tau + 34\tau^2 + 14\tau^3 + \tau^4)\\
c_1 &= 8\tau^3(3 + 28\tau + 34\tau^2 + 28\tau^3 + 3\tau^4)\\
c_2 &= -8\tau^6(1 - 52\tau - 90\tau^2 - 52\tau^3 + \tau^4)\\
c_3 &= (1 + 6\tau + \tau^2)^2.
\end{aligned}$$

*The divisor $Z(\mu(\tau))$ has a singularity of type $A_5$ at $(R, (1 : 1))$.*

[1] Note that the same heuristic reasoning might be applied at the point $(R, (-1 : 1))$, which is a point on $H$; the intersection number $H.Z(\lambda) = 8$ implies that $Z(\lambda)$ can only pass through this point when $H \subseteq \mathrm{Supp}(Z(\lambda))$, i.e., when $\lambda \in \mathrm{Span}(\{\lambda_0, \lambda_1, \lambda_2\})$. Investigation of this case did not yield surfaces with Picard numbers as large as those in Propositions 6.1 and 6.2.

While the proof of this theorem only requires a straightforward verification, see Remarks 5 and 6 for two ways to derive the coefficients $c_i$.

*Proof.* As $(R, (1:1))$ is not a base point of $|\Lambda|$, there is a proper subspace $\mathcal{H} \subseteq \Lambda$ describing those elements that vanish at $(R, (1:1))$; explicitly, if $\lambda = \sum_{i=0}^{3} a_i \lambda_i$ then $\mathcal{H}$ is described by

$$\mathcal{H}: \quad 4\tau^3 a_0 + a_2 - 8\tau^5(2 - \tau + 2\tau^2)a_3 = 0. \tag{5.5}$$

Putting $Z = Z_0/Z_1$, local parameters at $(R, (1:1))$ are given by $(u, v) = (x - 4\tau^3, Z - 1)$. If $\lambda \in \mathcal{H}$ then (upon expanding $\lambda_0, \ldots, \lambda_3$) we obtain in terms of these parameters the series expansion

$$\begin{aligned}\lambda &= a_0(\lambda_0 - 4\tau^3\lambda_2) + a_1\lambda_1 + a_3(\lambda_3 + 8\tau^5(2 - \tau + 2\tau^2)\lambda_2) \\ &= \left(\frac{1}{2\tau^3}a_0 - \frac{2(1-\tau)^2}{\tau}a_3\right)u^2 \\ &\quad + \big(10a_0 + 4a_1 - 48\tau^2(1 - \tau + \tau^2)a_3\big)uv \\ &\quad + \big(48\tau^3 a_0 + 32\tau^3 a_1 - 8\tau^4(6 + 34\tau + 7\tau^2 + 34\tau^3 + 6\tau^4)a_3\big)v^2 \\ &\quad + (\text{higher terms})\end{aligned} \tag{5.6}$$

This shows that in fact *every* $Z(\lambda) \in |\mathcal{H}|$ has a singularity at $(R, (1:1))$, which is isolated with multiplicity 2; for generic $\lambda \in \mathcal{H}$, it will be of type $A_1$.

Now consider the element $\mu(\tau)$. One may easily check that $(c_0, c_1, c_2, c_3)$ satisfies (5.5), and thus the local form of $\mu(\tau)$ near $(R, (1:1))$ will look like (5.6) with $a_i = c_i$. In this local form, one may also check that the hessian determinant at $(u, v) = (0, 0)$ vanishes, so there is a singularity of type $A_n$ for $n > 1$. The value of $n$ may be determined by repeated blow ups. Indeed, blowing up once reveals that the proper transform of the curve $Z(\mu(\tau))$ has one singularity where it meets the exceptional divisor, and blowing up a second time reveals that the proper transform has an $A_1$-singularity where it meets the (second) exceptional divisor. Hence the original singularity was of type $A_5$. See [Lyo] for details of this calculation in `Mathematica`. □

**Corollary 5.4.** *We have the following lower bounds for Picard numbers:*

1. *The Picard number of $Y(\mu(\tau))$ over $k$ (resp. $k(\sigma)$) is at least 7 (resp. at least 12).*
2. *The Picard number of the surface $X^\varepsilon(\mu(\tau))$ over $k$ is at least 7.*

*Proof.* The class of the canonical divisor and an Albanese fiber guarantee these surfaces have Picard number at least 2 over $k$. By Theorem 5.3, $Y_0(\mu(\tau))$ has $A_5$-singularities at the points above $(\pm R, (1:1)) \in \mathbf{B}|_U$, and this pair of points (which we recall are defined over $k(\sigma)$, see (5.4)) forms a single orbit under the actions of both $\iota$ and $\mathrm{Gal}(\bar{k}/k)$. Hence:

- On the resolution $Y(\mu(\tau))$, there will be two corresponding chains $C_1, \ldots, C_5$ and $C'_1, \ldots, C'_5$ consisting of $(-2)$-curves that are defined over $k(\sigma)$ and are exchanged by $\iota$. The divisors $C_i + C'_i$ will be invariant under $\mathrm{Gal}(\bar{k}/k)$, while the divisors $C_i - C'_i$ will be anti-invariant. This gives the first statement.

- The quotient surface $X_0^\varepsilon(\mu(\tau)) = Y_0(\mu(\tau))/\iota^\varepsilon$ will have a $k$-rational $A_5$-singularity, and the exceptional curves that this contributes to the Néron-Severi group of $X^\varepsilon(\mu(\tau))$ will guarantee that the latter surface has Picard number at least 7 over $k$. (Note that these exceptional curves pull back to the $\iota^\varepsilon$-invariant divisors $C_i + C_i'$ on $Y(\mu(\tau))$.)

□

*Remark* 5. The proof of Theorem 5.3 indicates one method for deriving the coefficients $c_i$ of $\mu(\tau)$. First one can compute the hessian determinant in the local equation (5.6) to obtain, in terms of the variables $a_0, a_1, a_3, \tau$, a smooth quadric curve $\mathcal{C}_2 \subseteq |\mathcal{H}|$ representing divisors that have a degenerate singularity at $(R, (1:1))$. If one blows up $\mathbf{B}$ at $(R, (1:1))$ and computes the type of singularity on the proper transform, then for general $\lambda \in \mathcal{C}_2$ one finds an $A_1$-singularity; that is, when one examines the local equation of the proper transform near its singularity, the hessian determinant does not vanish for a general element of $\mathcal{C}_2$. Thus the general element of $\mathcal{C}_2$ has an $A_3$-singularity at $(R, (1:1))$. However, if this determinant does vanish, then the singularity will be of higher order; this determinant defines a quintic curve $\mathcal{C}_5 \subseteq |\mathcal{H}|$ in the variables $a_0, a_1, a_3, \tau$. Hence one can focus upon values of $\lambda$ coming from points of the scheme $\mathcal{C}_2 \cap \mathcal{C}_5 \subseteq \mathbb{P}^2 \times \mathbb{A}^1_\tau$.

Ignoring cases when $H \subseteq \mathrm{Supp}(Z(\lambda))$, one may restrict to the open set where $a_3 = 1$. The radical decomposition for the ideal in $\mathbb{Q}[a_0, a_1, \tau]$ defining $\mathcal{C}_2 \cap \mathcal{C}_5$ on this open set can be computed quickly in `Magma`. (In fact, computing a probable radical decomposition is even quicker and usually suffices for this sort of experimentation.) Among the components that are identified by this decomposition, one of them corresponds precisely to the family $\mu(\tau)$.

See [Lyo] for an implementation of this approach.

*Remark* 6. The heuristic reasoning preceding Theorem 5.3 and the method in Remark 5 reflect the benefit of hindsight. In fact, we originally stumbled upon $\mu(\tau)$ by experimental analysis of the locus of singular elements in $|\Lambda|$. This locus of singular elements in $|\Lambda|$ is represented by a reducible hypersurface $\Sigma$ in $\mathbb{P}^3$. If one wishes to find $\lambda \in \Lambda$ with "richer" singularities, one well-known tactic is to seek points $(a_0 : a_1 : a_2 : a_3)$ where $\Sigma$ is highly singular, and then investigate $Z(\sum a_i \lambda_i)$.

In practice, however, we found that the task of determining equations for $\Sigma$ was only computationally feasible when we specialized $\tau$ to some specific value $\tau_0$ and worked in a prime characteristic $p$. After trying multiple pairs $(\tau_0, p)$, we came to believe in the existence of the noteworthy family $\mu(\tau)$, but did not have the equations in Theorem 5.3. By finessing the computations, we were able to determine:

(1) The coefficients of $\mu(\tau)$ in characteristic $p$ when one fixes the prime $p$.
(2) The coefficients of $\mu(\tau_0)$ in characteristic 0 when one fixes the value of $\tau_0$.

From (1) we learned the degrees of the coefficients $c_i(\tau)$ in Theorem 5.3. Then by choosing multiple values of $\tau_0$ in (2), we were able to use basic polynomial interpolation to fully determine $c_i(\tau)$.

## 6. Increasing the Picard number

By construction, for each $\lambda \in |\Lambda|$ the divisor $Z(\lambda)$ either contains $H = Z(Z_0+Z_1)$ in its support or it intersects $H$ in precisely the four points in (5.2), with each intersection having with multiplicity 2. If $\lambda = \sum a_i\lambda_i$ and one uses the polynomials in Proposition 4.1 to write down $Z(\lambda)\cap\mathbf{B}_U$ (so that one is working with polynomials in $k(E_\tau)[Z_0, Z_1]$), then this is reflected algebraically by the factorization

$$\lambda(Z_0, -Z_0) = a_3\lambda_3(Z_0, -Z_0) = \frac{a_3 Z_0^6(x-4\tau^2)^2(x-4\tau^4)^2}{x^2}$$

Hence if $a_3 = b_3^2 \in (k^\times)^2$ is a nonzero square in $k$, the divisor $\rho^*H$ in the double cover $\rho : Y_0(\lambda) \to \mathbf{B}$ will be reducible over $k$. If, as in (4.3), we identify $Y_0(\lambda)|_U$ with the hypersurface

$$Z\big(T^2 - \lambda(Z_0, Z_1)\big) = Z\Big(T^2 - \sum_{i=0}^{3} a_i\lambda_i(Z_0, Z_1)\Big)$$

in $U \times \mathbb{P}^2(1,1,3)$, then the divisor $\rho^*H|_U$ is the sum of the two curves

$$C = Z\left(T - \frac{b_3 Z_0^3(x-4\tau^2)(x-4\tau^4)}{x}\ ,\ Z_0+Z_1\right)$$

and

$$C' = Z\left(T + \frac{b_3 Z_0^3(x-4\tau^2)(x-4\tau^4)}{x}\ ,\ Z_0+Z_1\right).$$

But on $\mathbf{B}|_U$ the involutions $\iota^+, \iota^-$ take the form

$$\begin{aligned}\iota^+((x,y),(Z_0:Z_1:T)) &= \left(\left(\frac{16\tau^6}{x}, -\frac{16\tau^6 y}{x^2}\right), (Z_1:Z_0:T)\right)\\ \iota^-((x,y),(Z_0:Z_1:T)) &= \left(\left(\frac{16\tau^6}{x}, -\frac{16\tau^6 y}{x^2}\right), (Z_1:Z_0:-T)\right),\end{aligned}$$

and hence $(\iota^+)^*$ swaps $C$ and $C'$, while $(\iota^-)^*$ preserves them. (Note that we are using the fact that $H$ does not contain $\mathbf{B}_{P_0}$ or $\mathbf{B}_{P_1}$ as components, and hence the action of these involutions of $C, C'$ may be inferred by what happens on $\mathbf{B}|_U$.) Thus we have:

**Proposition 6.1.** *The following hold:*

(1) *For generic $\lambda \in |\Lambda|$, the surface $X^-(\lambda)$ has smooth canonical model (i.e., $Z(\lambda)$ is smooth) and geometric Picard number at least* 3.
(2) *The Picard number of $Y(\mu(\tau))$ over $k$ (resp. $k(\sigma)$) is at least* 8 *(resp. at least* 13*).*
(3) *The Picard number of $X^-(\mu(\tau))$ over $k$ is at least* 8.

One might wish to increase the Picard number of $X^\varepsilon(\mu(\tau))$ even further by searching for more singularities on the divisor $Z(\mu(\tau))$. A computer search reveals that this is possible only for a handful of values of $\tau$:

**Proposition 6.2.** *Let $\tau \in k$ satisfy the hypotheses of Theorem 5.3, and define*

$$\begin{aligned}f(X) \;=\; & 8X^{11} + 248X^{10} + 2868X^9 + 16304X^8 + 48479X^7\\ & +73647X^6 + 53611X^5 + 20851X^4 + 4565X^3 + 565X^2 + 37X + 1.\end{aligned}$$

(1) *Suppose that $f(\tau) = 0$. The singular locus of $Z(\mu(\tau))$ consists of the $A_5$-singularities from Theorem 5.3 and two additional $\iota$-conjugate $A_1$-singularities. Thus:*
  (a) *The geometric Picard number of $X^+(\mu(\tau))$ is at least 8.*
  (b) *The geometric Picard number of $X^-(\mu(\tau))$ is at least 9.*

(2) *Suppose that $f(\tau) \neq 0$. The singular locus of $Z(\mu(\tau))$ consists only of the $A_5$-singularities from Theorem 5.3.*

*In either case, the fibration $Y(\mu(\tau)) \to E_\tau$ is relatively minimal and thus the surfaces $X^+(\mu(\tau))$ and $X^-(\mu(\tau))$ are minimal surfaces with invariants $p_g = q = 1, K^2 = 2$.*

*Proof.* Recall that the $A_5$-singularities in Theorem 5.3 occur at $(\pm R, (1 : 1))$, where $x(\pm R) = 4\tau^3$. When one uses `Magma` to search for values of $\tau$ such that $Z(\mu(\tau))$ has a singularity above a point $P = (x, y) \in U$ with $x \neq 4\tau^3$, the conclusion is that $\tau$ must be a root of $f(X)$. On the other hand, if one searches for $\tau$ having singularities at points $(\pm R, (Z_0 : Z_1)) \in \mathbf{B}|_U$ with $(Z_0 : Z_1) \neq (1 : 1)$, no values of $\tau$ are returned. Moreover, one checks that $Z(\mu(\tau))$ is always smooth at the fiber $\mathbf{B}_{P_0}$ (and hence also $\mathbf{B}_{P_1}$). Hence (2) holds.

Now assume that $f(\tau) = 0$. From Theorem 5.3 and the previous paragraph, we know that the singular locus of $Z(\mu(\tau))$ contains at least four points. The polynomial $f$ is irreducible and the coefficients of $Z(\mu(\tau))$ belong to $\mathbb{Z}[\tau]$, so it suffices to show that (1) holds when $k = \mathbb{Q}(\tau)$. One can easily find a prime $p$ such that $\bar{f} \in \mathbb{F}_p[X]$ has a simple root $\bar{\tau}_p \in \mathbb{F}_p$, and by Hensel's lemma this lifts to a root $\tau_p \in \mathbb{Z}_p$; hence there is an embedding

$$\mathbb{Z}[\tau] \hookrightarrow \mathbb{Z}_p, \qquad \tau \mapsto \tau_p,$$

so that $Z(\mu(\tau))$ has an integral model over $\mathbb{Z}_p$. It is also easy to find $p$ having the additional property that $E_{\bar{\tau}_p} := E_\tau \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ (coming from $\tau \mapsto \tau_p \mapsto \bar{\tau}_p$) is smooth and $Z(\mu(\bar{\tau}_p)) := Z(\mu(\tau)) \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ has only two $A_5$- and two $A_1$-singularities. This implies that the same is true of $Z(\mu(\tau))$, and thus the first part of (1) holds. The statements about Picard numbers follow from Corollary 5.4 and Proposition 6.1.

To finish the proposition, we must show that $Y(\mu(\tau)) \to E_\tau$ is relatively minimal (so that we may then apply Theorem 2.6). Suppose that $C \subseteq Y(\mu(\tau))$ is a $(-1)$-curve that maps to a point $P \in E_\tau$. First note that the fibration map $Y(\mu(\tau)) \to E_\tau$ factors as

$$Y(\mu(\tau)) \longrightarrow Y_0(\mu(\tau)) \longrightarrow E_\tau$$

and the resolution map $Y(\mu(\tau)) \to Y_0(\mu(\tau))$ does not contract any $(-1)$-curves (since the branch curve $Z(\mu(\tau))$ has only simple singularities). Hence the image of $C$ in $Y_0(\mu(\tau))$ is a rational curve $C'$ belonging to the fiber over $P$. Given that the general fiber of $Y_0(\mu(\tau))$ is a smooth curve of genus 2, the fiber of $Y_0(\mu(\tau))$ must therefore be singular. There are two possibilities to consider:

- If the surface $Y_0(\mu(\tau))$ does not have a singularity above $P$, then $C$ does not meet the exceptional locus the resolution $Y(\mu(\tau)) \to Y_0(\mu(\tau))$, which implies $C' \simeq C$ is also a $(-1)$-curve. But the fiber of $Y_0(\mu(\tau))$ over $P$

can only contain a smooth rational curve if it is the union of two smooth rational curves that are exchanged by the hyperelliptic involution. Since the fibers of $Y(\mu(\tau))$ and $Y_0(\mu(\tau))$ over $P$ are isomorphic, all of this would imply that in fact $C^2 = -3$. This rules out this case.

- If $Y_0(\mu(\tau))$ is singular at some point of the fiber over $P$, then $Z(\mu(\tau))$ has a singularity over $P$. From the first half of the proposition (already proved above), this means this singularity is either one of the $A_5$-singularities described in Theorem 5.3, or $f(\tau) = 0$ and the singularity is one of the $A_1$-singularities described above. One can check, however, that in either of these cases, the fiber of $Y_0(\mu(\tau))$ is irreducible of positive geometric genus, and so cannot contain the curve $C'$ as above.

Hence $Y_0(\mu(\tau))$ must be minimal.

See [Lyo] for computational details relating to this proof. □

## 7. Picard numbers over finite fields

In Proposition 6.1 it was shown that the Picard number of $X^-(\mu(\tau))$ is at least 8, and in this section we show that this lower bound is an equality when $\tau = 3$. Following a method developed in [LL], this is achieved by determining the zeta function of the reduction of $Y(\mu(3))$ to $\mathbb{F}_{11}$, and using this to deduce the zeta function of the reduction of $X^-(\mu(3))$ over $\mathbb{F}_{11}$, which in turn gives an upper bound of 8 for the Picard number of $X^-(\mu(3))$ over $\mathbb{Q}$.

We discuss the general idea of the calculation, but refer to [LL, §4] for more details since the procedure is largely the same. The varieties $E_3$ and $Y(\mu(3))$ are defined above using polynomials with integral coefficients, and we use these equations to give us integral models for each of them. One may check, upon reducing modulo $p = 11$, that $E_3$ is smooth and $Z(\mu(3))$ has a singular locus consisting of two $A_5$-singularities. To ease notation, in this section we will write:

$$E := E_3, \quad Y := Y(\mu(3)), \quad Z := Z(\mu(3)), \quad X^+ := X^+(\mu(3)), \quad X^- := X^-(\mu(3)),$$

and for a $\mathbb{Z}$-scheme $\mathcal{S}$ we will let $\bar{\mathcal{S}} := \mathcal{S} \otimes_{\mathbb{Z}} \mathbb{F}_{11}$ denote its reduction modulo 11.

We now describe the zeta function of $\bar{Y}$. Given that $Z$ and $\bar{Z}$ have the same singularity types, it follows that the reduction $\bar{Y}$ of $Y$ is smooth. Hence the Betti numbers of $\bar{Y}$ (in $\ell$-adic cohomology) are the same as those of $Y$. From Proposition 6.2, $X^+$ is a smooth minimal surface with $p_g = q = 1$ and $K^2 = 2$, and so (arguing as in the proof of Theorem 2.6) the Betti numbers of $Y$ are

$$b_0(\bar{Y}) = b_4(\bar{Y}) = 1, \quad b_1(\bar{Y}) = b_3(\bar{Y}) = 2, \quad b_2(\bar{Y}) = 22.$$

One also computes the zeta function of $\bar{E} = \mathrm{Alb}(\bar{Y})$ to be

$$Z(\bar{E}, t) = \frac{1 + 11t^2}{(1-t)(1-11t)}.$$

Finally, the point $\bar{R} \in \bar{E}$ is defined only over $\mathbb{F}_{11^2}$, and hence the same is true of the exceptional curves arising from the two $A_5$-singularities of $\bar{Z}$. Putting this together with Proposition 6.1, we conclude that the zeta function of $\bar{Y}$ has the form

$$Z(\bar{Y},t) = \frac{(1+11t^2)(1+1331t^2)}{(1-t)(1-14641t)P_2(\bar{Y},t)},$$

where

$$P_2(\bar{Y},t) = (1-11t)^8(1+11t)^5 Q(t)$$

for some $Q \in \mathbb{Z}[t]$ of degree 9.

We can also determine the form of the closely-related zeta function of $\bar{X}^\varepsilon$. Using Corollary 5.4 and Proposition 6.1, along with the fact that $\mathrm{Alb}(X^\varepsilon) = \hat{E}$ is isogenous to $E$, we may deduce that the zeta function of $\bar{X}^\varepsilon$ has the form

$$Z(\bar{X}^\varepsilon,t) = \frac{(1+11t^2)(1+1331t^2)}{(1-t)(1-14641t)P_2(\bar{X}^\varepsilon,t)}, \tag{7.1}$$

where

$$P_2(\bar{X}^-,t) = (1-11t)^8 Q_-(t) \tag{7.2}$$

and

$$P_2(\bar{X}^+,t) = (1-11t)^7 Q_+(t), \tag{7.3}$$

for some $Q_\varepsilon \in \mathbb{Z}[t]$, with $\deg(Q_+) = 4$, and $\deg(Q_-) = 5$.

Since we do not possess equations for $\bar{X}^\varepsilon$, we cannot compute its zeta function by direct point counting. Instead, the key observation from [LL] is to use the following two facts, both of which result from $\bar{X}^\varepsilon$ being the quotient of $\bar{Y}$ by the free involution $\iota^\varepsilon$:

- The points in $\bar{X}^\varepsilon(\mathbb{F}_{11})$ are in bijection with (unordered) pairs $\{\mathcal{P}, \iota^\varepsilon(\mathcal{P})\}$ (with $\mathcal{P} \in \bar{Y}(\mathbb{F}_{11^2})$) that are preserved by the Frobenius map.
- The polynomial $P_2(\bar{X}^\varepsilon,t)$ must divide $P_2(\bar{Y},t)$ in $\mathbb{Z}[t]$.

Hence the general strategy is first list all points on $\bar{Y}(\mathbb{F}_{11^2})$ and analyze the $\iota^\varepsilon$-orbits to obtain the point count $\#\bar{X}^\varepsilon(\mathbb{F}_{11})$. Second, the Weil Conjectures allow us to compute $Z(\bar{Y},t)$ by counting points on $\bar{Y}(\mathbb{F}_{11^d})$ for sufficiently many $d$, and we use this to obtain several "candidates" for $Z(\bar{X}^\varepsilon,t)$. Finally, each of these candidates gives a prediction for the value $\#\bar{X}^\varepsilon(\mathbb{F}_{11})$ and, with enough luck, only one of the candidates will match the actual value. Here is what one finds upon carrying this out:

**Proposition 7.1.** *We have*

$$\begin{aligned} Q_-(t) &= 1+6t-99t^2+726t^3+14641t^4 \\ Q_+(t) &= (1+11t)(1-30t+429t^2-3630t^3+14641t^4). \end{aligned}$$

*Proof.* With the help of `Magma`, we do the following:

- By analyzing the $\iota^\varepsilon$ and Frobenius orbits of $\bar{Y}(\mathbb{F}_{11^2})$, we find

$$\#\bar{X}^-(\mathbb{F}_{11}) = 218, \qquad \#\bar{X}^+(\mathbb{F}_{11}) = 204$$

- The factor $P_2(\bar{Y}, t)$ in the zeta function of $\bar{Y}$ is

$$\begin{aligned} P_2(\bar{Y}, t) &= (1-11t)^8(1+11t)^6 \\ &\quad \cdot (1+6t-99t^2+726t^3+14641t^4) \\ &\quad \cdot (1-30t+429t^2-3630t^3+14641t^4), \end{aligned}$$

  where each factor of degree 4 is irreducible over $\mathbb{Q}$.

Hence each $Q_\varepsilon(t)$ must divide

$$Q(t) = (1+11t)(1+6t-99t^2+726t^3+14641t^4)(1-30t+429t^2-3630t^3+14641t^4).$$

Since $P_2(\bar{X}^-, t)$ divides $P_2(\bar{Y}, t)$, we use (7.2) to arrive at 3 possibilities for $P_2(\bar{X}^-, t)$:

- $P_2(\bar{X}^-, t) = (1-11t)^8(1-30t+429t^2-3630t^3+14641t^4)$
- $P_2(\bar{X}^-, t) = (1-11t)^8(1+6t-99t^2+726t^3+14641t^4)$
- $P_2(\bar{X}^-, t) = (1-11t)^8(1+11t)^4$

Since

$$\log Z(\bar{X}^-, t) = \sum_{d=1}^{\infty} \left(\#\bar{X}^-(\mathbb{F}_{11^d})\right)\frac{T^d}{d},$$

we then plug each of these possibilities into (7.1) to obtain a candidate for $Z(\bar{X}^-, t)$, compute the power series expansion of $\log Z(\bar{X}^-, t)$, and check whether the coefficient of $T$ is 218. This turns out to happen only for the second possibility.

The possibilities for $P_2(\bar{X}^+, t)$ include all of the three possibilities listed above for $P_2(\bar{X}^+, t)$ as well as the following additional three:

- $P_2(\bar{X}^+, t) = (1-11t)^7(1+11t)(1-30t+429t^2-3630t^3+14641t^4)$
- $P_2(\bar{X}^+, t) = (1-11t)^7(1+11t)(1+6t-99t^2+726t^3+14641t^4)$
- $P_2(\bar{X}^+, t) = (1-11t)^7(1+11t)^5$

Of the six possibilities, the only one that predicts $\#\bar{X}^+(\mathbb{F}_{11}) = 204$ is

$$P_2(\bar{X}^+, t) = (1-11t)^7(1+11t)(1-30t+429t^2-3630t^3+14641t^4).$$

□

**Corollary 7.2.** *The Picard numbers of $X^-$ over $k$ and $\bar{k}$ both equal 8. The Picard numbers of $X^+$ over $k$ and $\bar{k}$ are either 7 or 8.*

*Proof.* The geometric Picard number of $X^-$ is bounded above by the geometric Picard number of its reduction $\bar{X}^-$, which in turn is bounded above by the number of roots of $P_2(\bar{X}^-, t)$ that are of the form $\frac{1}{11}\zeta$, for a root of unity $\zeta$. This latter upper bound equals 8, given that the quartic polynomial $Q_-(t/11)$ is not cyclotomic. Hence Proposition 6.1 gives the result for $X^-$.

Likewise, we learn from $P_2(\bar{X}^+, t)$ that the geometric Picard number of $X^+$ is at most 8, and we may apply Corollary 5.4 to obtain the second statement. □

*Remark* 7. Lacking any evidence to the contrary, it seems reasonable to believe that the Picard number of $X^+$ is 7. However, a simple application of the Weil Conjectures shows that the number of roots $P_2(\bar{X}^+, t)$ of the form $\frac{1}{11}\zeta$ has the same parity as $b_2(X^+) = 12$, and hence 8 is the best possible upper bound one can obtain via the method in the proof of Corollary 7.2. The same sort of parity issue arises when trying to verify that a given K3 surfaces has an odd Picard number, and more involved characteristic $p$ methods (such as those in [vL, EJ]) have been developed to handle some of those cases. It might be possible to adapt one of those methods for use on $X^+$, but we have not attempted this.

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California State University, Fullerton, Department of Mathematics, 800 N. State College Blvd, Fullerton, CA 92834

*Email address*: `clyons@fullerton.edu`