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On solvable Leibniz algebras whose nilradical has characteristic sequence (m_1, m_2, m_3).

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Abstract. In this paper we classify solvable Leibniz algebras with characteristic sequence (m_1, m_2, m_3) and maximal possible dimension of complementary subspace. Additionally we show the triviality of the first group of cohomologies.

Keywords: Leibniz algebra, derivation, nilpotent algebras, nilradical, solvable Leibniz algebras

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1 Introduction

Leibniz algebras are relatively new, and have played a significant part in mathematics in the recent years. They were introduced by J.-L. Loday [15] as a generalization of Lie algebras. A finite-dimensional Leibniz algebra is a semisimple Lie algebra adjoined by a semidirect sum with a solvable radical [5]. Solvable can be divided into two parts, nilpotent and non-nilpotent. For solvable Leibniz algebras there are many classifications for various types of nilradicals, including Heisenberg nilradical [6], nul-filiform [9], filiform [8], naturally graded filiform [2, 10, 14], direct sum of null-filiform algebras [13], quasi-filiform [1, 7, 12], and trianglular [11]. We will consider the non-nilpotent Leibniz algebras whose nilradical has characteristic sequence (m_1, m_2, m_3) and complementary subspace has maximal possible dimension.

Our inspiration for this paper is derived from a paper which describes solvable Lie algebras which nilradical N and $C(N) = (m_1, m_2, \dots, m_k)$ for an arbitrary k [3]. We have begun to extend this case for specific Leibniz algebras. Actually, we focus on Leibniz algebras with a nilradical N has characteristic sequence (m_1, m_2, m_3). Our main results give the classification of the type of these algebras and the description of some its properties. Namely, we have that the first group of cohomologies of such algebras are trivial, and that said algebras are centerless. These results allow us to hope that the general case, as in [3], can be reached for Leibniz algebras case too.

In this paper we have finite-dimensional algebras and vector spaces over the field of complex numbers. It is to be assumed that omitted products are equal to zero and any solvable algebra is not nilpotent unless otherwise noted.

2 Preliminaries

In this section we give definitions and preliminary results that are necessary in this paper, also found in [1, 9, 15].

Definition 2.1. A vector space with a bilinear bracket $(L, [-,-])$ over a field $\mathbb{F}$ is called a Leibniz algebra if for any $x, y, z \in L$ the Leibniz identity

$$[[x, y], z] = [x, [y, z]] - [y, [x, z]].$$

From the Leibniz identity, we have that the elements $[x, x], [x, y] + [y, x]$ for any $x, y \in L$ are in $Ann_r(L) = x \in L \mid [y, x] = 0$ for all $y \in L$, where $Ann_r(L)$ is referred to the right annihilator of L, additionally, we have that $Ann_r(L)$ is a two-sided ideal of L. We also have another two-sided ideal $Center(L) = x \in L \mid [x, y] = [y, x] = 0,$ for all $y \in L$, which is the center of L.

Definition 2.2. A linear map $d : L \to L$ of a Leibniz algebra $(L, [-,-])$ is said to be a derivation if for all $x, y \in L$, the following condition holds:

$$d([x, y]) = [d(x), y] + [x, d(y)].$$

For a given element x of a Leibniz algebra L, the right multiplication operators $\mathcal{R}_x : L \to L,\ \mathcal{R}_x(y) = [y, x],\ y \in L$ are derivations (similarly, for a left Leibniz algebra, the left multiplication operators $\mathcal{L}_x : L \to L,\ \mathcal{L}_x(y) = [x, y], y \in L$, are derivations). These are referred to as inner derivations and denoted by $Inner(L)$. All other derivations are said to be outer. $\mathcal{R}(L)$ has a structure of Lie algebra by way. of the bracket $[\mathcal{R}_x, \mathcal{R}_y] = \mathcal{R}_x\mathcal{R}_y - \mathcal{R}_y\mathcal{R}_x = \mathcal{R}_{[y,x]}$. Additionally there exists an antisymmetric isomorphism between $\mathcal{R}(L)$ and the quotient algebra $L/Ann_r(L)$.

Definition 2.3. A Leibniz algebra L is said to be nilpotent (respectively, solvable), if there exists $n \in \mathbb{N}$ $(m \in \mathbb{N})$ such that $L^n = 0$ (respectively, $L^{[m]} = 0$).

Definition 2.4. The nilradical of a Leibniz algebra is defined to be the maximal nilpotent ideal of the algebra.

For an element of $L \setminus L^2$ we consider the Jordan form of the linear map,

$$R_x = \begin{pmatrix} \mathcal{J}_{m_1} & 0 & \dots & 0 \\ 0 & \mathcal{J}_{m_2} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \mathcal{J}_{m_k} \end{pmatrix}$$

We define $c(x) = (n_1, n_2, ..., n_k)$ where $n_1 \geq n_2 \geq \cdots \geq n_k$ for some $k \in \mathbb{N}$. The characteristic sequence of L is defined to be

$$C(L) = \max C(x) \text{ for } x \in L \setminus L^2.$$

For the definitions and preliminary results on cohomologies of Leibniz algebras we refer to [4, 16]. Here we give only definition of HL^1. In fact,

$$HL^1 = Der(L)/Inner(L).$$

3 Main Results

We consider the Leibniz algebra with characteristic sequence $c(N) = (m_1, m_2, m_3)$ and basis $\{e_1, \dots, e_{m_1}, f_1, \dots, f_{m_2}, g_1, \dots, g_{m_3}\}$, where $m_1 \geq m_2 \geq m_3 > 1$. This algebra is defined by the following products:

$$\begin{cases} [e_i, e_1] = e_{i+1}, & \text{for } 1 \leq i \leq m_1 - 1, \\ [f_i, e_1] = f_{i+1}, & \text{for } 1 \leq i \leq m_2 - 1, \\ [g_i, e_1] = g_{i+1}, & \text{for } 1 \leq i \leq m_3 - 1. \end{cases}$$

Proposition 3.1. *Any derivation of this algebra have the following form:*

$$\begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}$$

where $A_1,\ A_2,\ A_3,\ B_1,\ B_2,\ B_3,\ C_1,\ C_2,\ C_3$ are block matrices defined as follows with dimensions, $m_1 \times m_1,\ m_1 \times m_2,\ m_1 \times m_3,\ m_2 \times m_1,\ m_2 \times m_2,\ m_2 \times m_3,\ m_3 \times m_1,\ m_3 \times m_2,\ m_3 \times m_3$, respectively.

$$A_1 = \sum_{i=1}^{m_1} i\alpha_{1,1} E_{i,i} + \sum_{i=1}^{m_1-1} \sum_{t=1}^{m_1-i} \alpha_{1,t+1} E_{i,i+t}, \quad A_2 = \sum_{i=1}^{m_2} \sum_{t=0}^{m_2-i} \alpha_{2,t+1} E_{i,i+t},$$

$$A_3 = \sum_{i=1}^{m_3} \sum_{t=0}^{m_3-i} \alpha_{3,t+1} E_{i,i+t},$$

$$B_1 = \sum_{i=1}^{m_2} \sum_{t=1}^{m_2-i+1} \beta_{1,m_1-m_2+t} E_{i,m_1-m_2+t}, \quad B_3 = \sum_{i=1}^{m_3} \sum_{t=0}^{m_3-i} \beta_{3,t+1} E_{i,i+t},$$
$$B_2 = \sum_{i=1}^{m_2} ((i-1)\alpha_{1,1} + \beta_{2,1}) E_{i,i} + \sum_{i=1}^{m_2} \sum_{t=1}^{m_2-i} \beta_{2,t+1} E_{i,i+t},$$

$$C_1 = \sum_{i=1}^{m_3} \sum_{t=1}^{m_3-i+1} \gamma_{1,m_1-m_3+t} E_{i,m_1-m_3+t}, \quad C_2 = \sum_{i=1}^{m_3} \sum_{t=1}^{m_3-i+1} \gamma_{2,m_2-m_3+t} E_{i,m_1-m_3+t},$$
$$C_3 = \sum_{i=1}^{m_3} ((i-1)\alpha_{1,1} + \gamma_{3,1}) E_{i,i} + \sum_{i=1}^{m_3} \sum_{t=1}^{m_3-i} \gamma_{3,t+1} E_{i,i+t}.$$

Proof. Let the following be the decomposition in the algebra basis of a derivation d on the generators of the algebra.

$$\begin{cases} d(e_1) = \sum\limits_{i=1}^{m_1} \alpha_{1,i} e_i + \sum\limits_{i=1}^{m_2} \alpha_{2,i} f_i + \sum\limits_{i=1}^{m_3} \alpha_{3,i} g_i, \\ d(f_1) = \sum\limits_{i=1}^{m_1} \beta_{1,i} e_i + \sum\limits_{i=1}^{m_2} \beta_{2,i} f_i + \sum\limits_{i=1}^{m_3} \beta_{3,i} g_i, \\ d(g_1) = \sum\limits_{i=1}^{m_1} \gamma_{1,i} e_i + \sum\limits_{i=1}^{m_2} \gamma_{2,i} f_i + \sum\limits_{i=1}^{m_3} \gamma_{3,i} g_i. \end{cases}$$

We proceed by using the property of a derivation to find constraints on the coefficients

$$\begin{aligned} d(e_2) &= d([e_1, e_1]) = [d(e_1), e_1] + [e_1, d(e_1)] \\ &= [\sum_{i=1}^{m_1} \alpha_{1,i} e_i + \sum_{i=1}^{m_2} \alpha_{2,i} f_i + \sum_{i=1}^{m_3} \alpha_{3,i} g_i, e_1] + \\ & [e_1, \sum_{i=1}^{m_1} \alpha_{1,i} e_i + + \sum_{i=1}^{m_2} \alpha_{2,i} f_i + \sum_{i=1}^{m_3} \alpha_{3,i} g_i] \\ &= 2\alpha_{1,1} e_2 + \sum_{i=3}^{m_1} \alpha_{1,i-1} e_i + \sum_{i=2}^{m_2} \alpha_{2,i-1} f_i + \sum_{i=2}^{m_3} \alpha_{3,i-1} g_i. \end{aligned}$$

By induction and the derivation property, we derive

$$d(e_i) = d([e_{i-1}, e_1]) = d([e_{i-2}, e_1], e_1]) = d([e_3, e_1], e_1], ...], e_1]). \qquad (3.1)$$

From (3.1) we have equalities

$$\begin{cases} d(e_i) & = i\alpha_{1,1}e_i + \sum\limits_{j=i+1}^{m_1} \alpha_{1,j-i+1}e_j + \\ & + \sum\limits_{j=i}^{m_2} \alpha_{2,j-i+1}f_j + \sum\limits_{j=i}^{m_3} \alpha_{3,j-i+1}g_j, \ 1 \leq i \leq m_3, \\ d(e_i) & = i\alpha_{1,1}e_i + \sum\limits_{j=i+1}^{m_1} \alpha_{1,j-i+1}e_j + \sum\limits_{j=i}^{m_2} \alpha_{2,j-i+1}f_j, \ m_3+1 \leq i \leq m_2, \\ d(e_i) & = i\alpha_{1,1}e_i + \sum\limits_{j=i+1}^{m_1} \alpha_{1,j-i+1}e_j, \ m_2+1 \leq i \leq m_1. \end{cases}$$

Consider

$$0 = d([e_1, f_1]) = [d(e_1), f_1]+[e_1, d(f_1)] = [e_1, \sum_{i=1}^{m_1} \beta_{1,i}e_i+\sum_{i=1}^{m_2} \beta_{2,i}f_i+\sum_{i=1}^{m_3} \beta_{3,i}g_i] = \beta_{1,1}e_2.$$

Hence $\beta_{1,1} = 0$.

$$\begin{aligned} d(f_2) = d([f_1, e_1]) &= [d(f_1), e_1] + [f_1, d(e_1)] \\ &= [\sum_{i=2}^{m_1} \beta_{1,i}e_i + \sum_{i=1}^{m_2} \beta_{2,i}f_i + \sum_{i=1}^{m_3} \beta_{3,i}g_i, e_1] + [f_1, \sum_{i=1}^{m_1} \alpha_{1,i}e_i + \sum_{i=1}^{m_2} \alpha_{2,i}f_i + \sum_{i=1}^{m_3} \alpha_{3,i}g_i] \\ &= \sum_{i=3}^{m_1} \beta_{1,i-1}e_i + (\alpha_{1,1} + \beta_{2,1})f_2 + \sum_{i=3}^{m_2} \beta_{2,i-1}f_i + \sum_{i=2}^{m_3} \beta_{3,i-1}g_i. \end{aligned}$$

According to the equality (3.1) for basis elements f_i we get

$$\begin{cases} d(f_i) & = \sum\limits_{j=i+1}^{m_1} \beta_{1,j-i+1}e_j + ((i-1)\alpha_{1,1} + \beta_{2,1})f_i + \\ & + \sum\limits_{j=i+1}^{m_2} \beta_{2,j-i+1}f_j + \sum\limits_{j=i}^{m_3} \beta_{3,j-i+1}g_i, \quad 1 \leq i \leq m_3, \\ d(f_i) & = \sum\limits_{j=i+1}^{m_1} \beta_{1,j-i+1}e_j + ((i-1)\alpha_{1,1} + \beta_{2,1})f_i + \\ & \sum\limits_{j=i+1}^{m_2} \beta_{2,j-i+1}f_j, \quad m_3+1 \leq i \leq m_2. \end{cases}$$

From

$$0 = d([f_{m_2}, e_1]) = [d(f_{m_2}), e_1] + [f_{m_2}, d(e_1)] =$$

$$= [\sum_{j=m_2+1}^{m_1} \beta_{1,j-m_2+1}e_j + ((m_2-1)\alpha_{1,1} + \beta_{2,1})f_{m_2}, e_1] = \sum_{j=m_2+2}^{m_1} \beta_{1,j-m_2}e_j,$$

we obtain $\beta_{1,t} = 0,\ 2 \le t \le m_1 - m_2$.

Similarly to before, we conclude $\gamma_{1,1} = 0$. Applying the equality (3.1) for basis elements $g_i,\ 1 \le i \le m_3$ we have

$$d(g_i) = \sum_{j=i+1}^{m_1} \gamma_{1,j-i+1} e_j + \sum_{j=i}^{m_2} \gamma_{2,j-i+1} f_j + ((i-1)\alpha_{1,1} + \gamma_{3,1}) g_i + \sum_{j=i+1}^{m_3} \gamma_{3,j-i+1} g_j.$$

Analogously, we consider

$$0 = d([g_{m_3}, e_1]) = [\sum_{j=m_3+1}^{m_1} \gamma_{1,j-m_3+1} e_j + \sum_{j=m_3}^{m_2} \gamma_{2,j-m_3+1} f_j +$$

$$+ ((m_3-1)\alpha_{1,1} + \gamma_{3,1}) g_{m_3}, e_1] = \sum_{j=m_3+2}^{m_1} \gamma_{1,j-m_3} e_j + \sum_{j=m_3+1}^{m_2} \gamma_{2,j-m_3} f_j.$$

Thus, we have $\gamma_{1,t} = 0$ for $2 \le t \le m_1 - m_3$ and $\gamma_{2,t} = 0$ for $1 \le t \le m_2 - m_3$.

□

For solvable Leibniz algebras with nilradical N and dimension of complemented space of nilradical to an algebra is equal to s, we shall use the notation $R(N, s)$.

For a solvable Leibniz algebra R with nilradical $N = \{e_1, \dots, e_{m_1}, f_1, \dots, f_{m_2}, g_1, \dots, g_{m_3}\}$ we obtain a description of $R = N \oplus Q$ with dim $Q = 3$. Thus, we assume that $\{e_1, \dots, e_{m_1}, f_1, \dots, f_{m_2}, g_1, \dots, g_{m_3}, x_1, x_2, x_3\}$ be a basis of $R(N, 3)$.

Theorem 3.2. *An arbitrary algebra of the family $R(N, 3)$ is isomorphic to the following algebra:*

$$\begin{cases} [e_i, e_1] = e_{i+1}, & 1 \le i \le m_1 - 1, \quad [f_i, e_1] = f_{i+1}, \quad 1 \le i \le m_2 - 1, \\ [g_i, e_1] = g_{i+1}, & 1 \le i \le m_3 - 1, \quad [e_i, x_1] = i e_i, \quad 1 \le i \le m_1, \\ [f_i, x_1] = (i-1) f_i, & 2 \le i \le m_2, \quad [g_i, x_1] = (i-1) g_i, \quad 2 \le i \le m_3, \\ [f_i, x_2] = f_i, & 1 \le i \le m_2, \quad [g_i, x_3] = g_i, \quad 1 \le i \le m_3, \\ [x_1, e_1] = -e_1. \end{cases}$$

Proof. Consider the matrix from of a derivation from the case given in Proposition 3.1. Since parameters $\alpha_{1,1},\ \beta_{2,1},\ \gamma_{3,1}$ are in the diagonal, from [1, Theorem 3.9] we get only three nil-independent derivations which correspond to the values of $(\alpha_{1,1}, \beta_{2,1}, \gamma_{3,1})$ as $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$. We denote these derivations as $\mathcal{R}_{x_1}$, $\mathcal{R}_{x_2}$, $\mathcal{R}_{x_3}$ respectively. This gives us the following products,

$$\begin{cases} [e_i,x_1]=ie_i, & 1\leq i\leq m_1, \\ [f_i,x_1]=(i-1)f_i, & 2\leq i\leq m_2, \quad [g_i,x_1]=(i-1)g_i, \quad 2\leq i\leq m_3, \\ [f_i,x_2]=f_i, & 1\leq i\leq m_2, \quad [g_i,x_3]=g_i, \qquad\qquad 1\leq i\leq m_3. \end{cases}$$

We introduce denotations

$$\begin{cases} [x_i,e_1]=\sum\limits_{t=1}^{m_1} d_{i,1}^{1,t}e_t+\sum\limits_{t=1}^{m_2} d_{i,1}^{2,t}f_t+\sum\limits_{t=1}^{m_3} d_{i,1}^{3,t}g_t, \ 1\leq i\leq 3, \\ [x_i,f_1]=\sum\limits_{t=1}^{m_1} d_{i,2}^{1,t}e_t+\sum\limits_{t=1}^{m_2} d_{i,2}^{2,t}f_t+\sum\limits_{t=1}^{m_3} d_{i,2}^{3,t}g_t, \ 1\leq i\leq 3, \\ [x_i,g_1]=\sum\limits_{t=1}^{m_1} d_{i,3}^{1,t}e_t+\sum\limits_{t=1}^{m_2} d_{i,3}^{2,t}f_t+\sum\limits_{t=1}^{m_3} d_{i,3}^{3,t}g_t, \ 1\leq i\leq 3. \end{cases}$$

Taking into account that $[a,b]+[b,a]\in Ann_R(R)$ $\forall a,b\in R$ we deduce

$$\begin{array}{l} 0=[e_1,[x_1,e_1]+[e_1,x_1]]=d_{1,1}^{1,1}e_2+e_2=(d_{1,1}^{1,1}+1)e_2, \\ 0=[e_1,[x_i,e_1]+[e_1,x_i]]=d_{i,1}^{1,1}e_2, \quad 1\leq i\leq 2, \\ 0=[e_1,[x_i,f_1]+[f_1,x_i]]=d_{i,2}^{1,1}e_2, \quad 1\leq i\leq 3, \\ 0=[e_1,[x_i,g_1]+[g_1,x_i]]=d_{i,3}^{1,1}e_2, \quad 1\leq i\leq 3. \end{array}$$

Hence, $d_{1,1}^{1,1}=-1, \ d_{2,1}^{1,1}=d_{3,1}^{1,1}=d_{i,j}^{1,1}=0, \ 1\leq i\leq 3, \ 1\leq j\leq 2.$

We set

$$[x_i,x_j]=\sum_{t=1}^{m_1}\xi_{i,j}^t e_t+\sum_{t=1}^{m_2}\zeta_{i,j}^t f_t+\sum_{t=1}^{m_3}\mu_{i,j}^t g_t, \ 1\leq i,j\leq 3.$$

Consider the change of basis

$$x_i^{'}=x_i-\sum_{i=2}^{m_1} d_{i,1}^{1,t+1}e_i-\sum_{i=1}^{m_2} d_{i,1}^{2,t+1}f_i-\sum_{i=1}^{m_3} d_{i,1}^{3,t+1}g_i, \quad 1\leq i\leq 3.$$

Then

$$[x_1,e_1]=-e_1+d_{1,1}^{1,2}e_2+d_{1,1}^{2,1}f_1+d_{1,1}^{3,1}g_1,$$

$$[x_i,e_1]=d_{i,1}^{1,2}e_2+d_{i,1}^{2,1}f_1+d_{i,1}^{3,1}g_1, \ 1\leq i\leq 2.$$

Considering the Leibniz identity for the triples of elements $\{e_1, x_i, x_i\},\ 1 \leq i \leq 3$ we have $\xi^1_{i,i} = 0,\ 1 \leq i \leq 3.$

Next, we consider

$$0 = [[x_1, e_1], x_1] - [[x_1, x_1], e_1] - [x_1, [e_1, x_1]] = [-e_1 + d^{1,2}_{1,1}e_2+$$
$$+ d^{2,1}_{1,1}f_1 + d^{3,1}_{1,1}g_1, x_1] - [\sum_{t=2}^{m_1} \xi^t_{1,1}e_t + \sum_{t=1}^{m_2} \zeta^t_{1,1}f_t + \sum_{t=1}^{m_3} \mu^t_{1,1}g_t, e_1] - [x_1, e_1]$$
$$= -e_1 + 2d^{1,2}_{1,1}e_2 - \sum_{t=3}^{m_1} \xi^{t-1}_{1,1}e_t - \sum_{t=2}^{m_2} \zeta^{t-1}_{1,1}f_t -$$
$$- \sum_{t=2}^{m_3} \mu^{t-1}_{1,1}g_t + e_1 - d^{12}_{1,1}e_2 - d^{21}_{1,1}f_1 - d^{31}_{1,1}g_1$$
$$= -\sum_{t=3}^{m_1} \xi^{t-1}_{1,1}e_t - \sum_{t=2}^{m_2} \zeta^{t-1}_{1,1}f_t - \sum_{t=2}^{m_3} \mu^{t-1}_{1,1}g_t + d^{1,2}_{1,1}e_2 - d^{2,1}_{1,1}f_1 - d^{3,1}_{1,1}g_1.$$

From this we have

$$[x_1, e_1] = -e_1, \quad [x_1, x_1] = \xi^{m_1}_{1,1}e_{m_1} + \zeta^{m_2}_{1,1}f_{m_2} + \mu^{m_3}_{1,1}g_{m_3}.$$

We now consider the Leibniz identity for triple elements $\{x_2, e_1, x_2\},\ \{x_3, e_1, x_3\},\ \{x_i, e_1, x_j\},\ 1 \leq i \neq j \leq 3$. Then we obtain

$$\begin{array}{ll} d^{1,2}_{2,1} = \xi^1_{2,1},\ d^{1,2}_{3,1} = \xi^1_{3,1}, & d^{2,1}_{i,1} = d^{3,1}_{i,1} = 0,\ 2 \leq i \leq 3, \\ \xi^1_{1,i} = \xi^1_{i,j} = xi^t_{i,1} = \xi^t_{1,i} = \xi^t_{i,j} = 0, & 2 \leq i,j \leq 3,\ 2 \leq t \leq m_1 - 1, \\ \zeta^t_{1,i} = \zeta^t_{j,1} = \zeta^t_{i,j} = 0, & 2 \leq i,j \leq 3,\ 1 \leq t \leq m_2 - 1, \\ \mu^p_{1,i} = \mu^p_{j,1} = \mu^p_{i,j} = 0, & 2 \leq i,j \leq 3,\ 1 \leq p \leq m_3 - 1. \end{array}$$

Thus,

$$\begin{cases} [x_i, e_1] = d^{1,2}_{i,1}e_1, & 2 \leq i \leq 3, \\ [x_i, x_1] = d^{1,2}_{i,1}e_1 + \xi^{m_1}_{i,1}e_{m_1} + \zeta^{m_2}_{i,1}f_{m_2} + \mu^{m_3}_{i,1}g_{m_3}, & 2 \leq i \leq 3, \\ [x_i, x_j] = \xi^{m_1}_{i,j}e_{m_1} + \zeta^{m_2}_{i,j}f_{m_2} + \mu^{m_3}_{i,j}g_{m_3},\ 1 \leq i \leq 3, & 2 \leq j \leq 3. \end{cases}$$

From the Leibniz identity for triple elements $\{x_1, f_1, x_2\},\ \{x_i, f_1, x_1\},\ \{x_j, e_1, f_1\},\ 1 \leq i \leq 3,\ 2 \leq j \leq 3$ we deduce

$$[x_i, f_1] = 0 \quad 1 \leq i \leq 3.$$

Using the same arguments for g_1 we get $[x_i, g_1] = 0, \quad 1 \leq i \leq 3.$

In efforts to further simplify the products of $[x_i, x_j]$ for $1 \leq i, j \leq 3$ we use the Leibniz identity to the following products:

$$\begin{aligned} 0 =&[x_1, [x_1, x_2]] = [[x_1, x_1], x_2] - [[x_1, x_2], x_1] = \\ &= \zeta_{1,1}^{m_2} f_{m_2} - m_1 \xi_{1,2}^{m_1} e_{m_1} - (m_2 - 1)\zeta_{1,2}^{m_2} f_{m_2} - (m_3 - 1)\mu_{1,2}^{m_3} g_{m_3}. \end{aligned}$$

Consequently,

$$\zeta_{1,1}^{m_2} = (m_2 - 1)\zeta_{1,2}^{m_2}, \quad \xi_{1,2}^{m_1} = 0, \quad \mu_{1,2}^{m_3} = 0.$$

Hence, $[x_1, x_2] = \zeta_{1,2}^{m_2} f_{m_2} + \mu_{1,2}^{m_3} g_{m_3}$.

Consider

$$0 = [x_1, [x_2, x_3]] = [[x_1, x_2], x_3] - [[x_1, x_3], x_2] = \mu_{1,2}^{m_3} g_{m_3} - \zeta_{1,3}^{m_2} f_{m_2},$$

$$0 = [x_2, [x_3, x_2]] = [[x_2, x_3], x_2] - [[x_2, x_2], x_3] = \zeta_{2,3}^{m_2} f_{m_2} - \mu_{2,2} g_{m_3}.$$

Therefore, we get

$$\begin{gathered} [x_1, x_2] = \zeta_{1,2}^{m_2} f m_2, \quad [x_1, x_3] = \xi_{1,3}^{m_1} e_{m_1} + \mu_{1,3}^{m_3} g_{m_3}, \\ [x_2, x_3] = \xi_{2,3}^{m_1} e_{m_1} + \mu_{2,3}^{m_3}, \quad [x_2, x_2] = \xi_{2,2}^{m_1} e_{m_1} + \zeta_{2,2}^{m_2} f_{m_2}. \end{gathered}$$

Let us take the change of basis element x_i as follow

$$x_i' = x_i - \frac{\xi_{i,1}^{m_1}}{m_1 - 1} e_{m_1} - \frac{\zeta_{i,1}^{m_2}}{m_2 - 1} f_{m_2} - \frac{\mu_{i,1}^{m_3}}{m_3 - 1} g_{m_3}, \quad 1 \leq i \leq 3.$$

It is easy to see that under the above change we get $[x_i^{'}, x_1^{'}] = 0, \ 1 \leq i \leq 3$.

We now consider the Leibniz identity for triple elements $\{x_i, x_j, x_1\}, \ 1 \leq i \leq 3, \ 2 \leq j \leq 3$, then we deduce

$$[x_1, x_2] = [x_1, x_3] = [x_2, x_2] = [x_2, x_3] = [x_3, x_2] = [x_3, x_3] = 0.$$

So, we obtain $[x_i, x_j] = 0$ for $1 \leq i, j \leq 3$. □

In order to start the description we need to know the derivations of solvable Leibniz algebras $R(N, 3)$ whose nilradical has characteristic sequence (m_1, m_2, m_3).

Proposition 3.3. *Any derivations of the algebra $R(N,3)$ have the following matrix form:*

$$\begin{pmatrix} A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & C & 0 \\ D & 0 & 0 & 0 \end{pmatrix},$$

where A, B, C, D are block matrices of the sizes $m_1 \times m_1,\ m_2 \times m_2,\ m_3 \times m_3,\ 3 \times m_1$, respectively and defined as follows

$$A = \sum_{i=1}^{m_1} i\alpha_{1,1} E_{i,i} + \sum_{i=1}^{m_1-1} \alpha_{1,2} E_{i,i+1},$$

$$B = \sum_{i=1}^{m_2} ((i-1)\alpha_{1,1} + \beta_{2,1}) E_{i,i} + \sum_{i=1}^{m_2-1} \alpha_{1,2} E_{i,i+1},$$

$$C = \sum_{i=1}^{m_3} ((i-1)\alpha_{1,1} + \gamma_{3,1}) E_{i,i} + \sum_{i=1}^{m_3-1} \alpha_{1,2} e_{i,i+1}, \quad D = -\alpha_{1,2} E_{1,1}.$$

Proof. Let us introduce denotations

$$\begin{cases} d(e_1) = \sum\limits_{i=1}^{m_1} \alpha_{1,i} e_i + \sum\limits_{i=1}^{m_2} \alpha_{2,i} f_i + \sum\limits_{i=1}^{m_3} \alpha_{3,i} g_i + \alpha_4 x_1 + \alpha_5 x_2 + \alpha_6 x_3, \\ d(f_1) = \sum\limits_{i=1}^{m_1} \beta_{1,i} e_i + \sum\limits_{i=1}^{m_2} \beta_{2,i} f_i + \sum\limits_{i=1}^{m_3} \beta_{3,i} g_i + \beta_4 x_1 + \beta_5 x_2 + \beta_6 x_3, \\ d(g_1) = \sum\limits_{i=1}^{m_1} \gamma_{1,i} e_i + \sum\limits_{i=1}^{m_2} \gamma_{2,i} f_i + \sum\limits_{i=1}^{m_3} \gamma_{3,i} g_i + \gamma_4 x_1 + \gamma_5 x_2 + \gamma_6 x_3, \\ d(x_1) = \sum\limits_{i=1}^{m_1} \delta_{1i} e_i + \sum\limits_{i=1}^{m_2} \delta_{2i} f_i + \sum\limits_{i=1}^{m_3} \delta_{3,i} g_i + \delta_4 x_1 + \delta_5 x_2 + \delta_6 x_3, \\ d(x_2) = \sum\limits_{i=1}^{m_1} \xi_{1,i} e_i + \sum\limits_{i=1}^{m_2} \xi_{2,i} f_i + \sum\limits_{i=1}^{m_3} \xi_{3,i} g_i + \xi_4 x_1 + \xi_5 x_2 + \xi_6 x_3, \\ d(x_3) = \sum\limits_{i=1}^{m_1} \mu_{1,i} e_i + \sum\limits_{i=1}^{m_2} \mu_{2,i} f_i + \sum\limits_{i=1}^{m_3} \mu_{3,i} g_i + \mu_4 x_1 + \mu_5 x_2 + \mu_6 x_3. \end{cases}$$

We begin by considering the derivation property to obtain the description. First, we focus on products that will simplify the derivation of $d(x_1)$.

In fact, from

$$d(e_1) = d([e_1, x_1]) = [d(e_1), x_1] + [e_1, d(x_1)]$$

we get $d(e_1) = [d(e_1), x_1] + [e_1, d(x_1)]$.

Consider

$$\begin{aligned} 0 &= [d(e_1), x_1] + [e_1, d(x_1)] - d(e_1) \\ &= \sum_{i=3}^{m_1}(i-1)\alpha_{1,i}e_i + \sum_{i=3}^{m_2}(i-2)\alpha_{2,i}f_i + \sum_{i=3}^{m_3}(i-2)\alpha_{3,i}g_i + \\ &+ \delta_{1,1}e_2 + \delta_4 e_1 - \alpha_4 x_1 - \alpha_5 x_2 - \alpha_6 x_3. \end{aligned}$$

From the above computations, we obtain

$$\begin{aligned} d(e_1) &= \alpha_{1,1}e_1 + \alpha_{1,2}e_2 + \alpha_{2,2}f_2 + \alpha_{3,2}g_2, \\ d(x_1) &= \sum_{i=1}^{m_1}\delta_{1i}e_i + \sum_{i=1}^{m_2}\delta_{2i}f_i + \sum_{i=1}^{m_3}\delta_{3,i}g_i + \delta_5 x_2 + \delta_6 x_3. \end{aligned}$$

Notice that $[f_1, x_1] = 0$. Hence, $d([f_1, x_1]) = 0$.

We consider the following:

$$\begin{aligned} 0 = d([f_1, x_1]) &= \sum_{i=1}^{m_1} i\beta_{1,i}e_i + \delta_5 f_1 + (\beta_{1,1} + \delta_{1,1})f_2 + \\ &+ \sum_{i=3}^{m_2}(i-1)\beta_{2,i}f_i + \sum_{i=2}^{m_3}(i-1)\beta_{3,i}g_i. \end{aligned}$$

Thus, we now have the following conditions:

$$\begin{aligned} d(f_1) &= \beta_{2,1}f_1 + \alpha_{1,2}f_2 + \beta_{3,1}g_1 + \beta_4 x_1 + \beta_5 x_2 + \beta_6 x_3, \\ d(x_1) &= -\alpha_{1,2}e_1 + \sum_{i=2}^{m_1}\delta_{1,i}e_i + \sum_{i=1}^{m_2}\delta_{2,i}f_i + \sum_{i=1}^{m_3}\delta_{3,i}g_i + \delta_6 x_3. \end{aligned}$$

In a similar way we obtain

$$\begin{aligned} d(g_1) &= \gamma_{2,1}f_1 + \gamma_{3,1}g_1 + \alpha_{1,2}g_2 + \gamma_4 x_1 + \gamma_5 x_2 + \gamma_6 x_3, \\ d(x_1) &= -\alpha_{1,2}e_1 + \sum_{i=2}^{m_1}\delta_{1,i}e_i + \sum_{i=1}^{m_2}\delta_{2,i}f_i + \sum_{i=1}^{m_3}\delta_{3,i}g_i. \end{aligned}$$

Applying the derivation property to the products of the algebra $R(N,3)$ we derive

$$d(e_i) = i\alpha_{1,1}e_i + \alpha_{1,2}e_{i+1}, \quad 1 \le i \le m_1,$$
$$d(f_i) = ((i-1)\alpha_{1,1} + \beta_{2,1})f_i + \alpha_{1,2}f_{i+1}, \quad 1 \le i \le m_2,$$
$$d(g_i) = ((i-1)\alpha_{1,1} + \gamma_{3,1})g_i + \alpha_{1,2}g_{i+1}, \quad 1 \le i \le m_3.$$

□

Below we discuss the low level cohomology groups of $R(N,3)$.

Theorem 3.4. *$Center(R(N,3)) = 0$ and $HL^1(R(N,3), R(N,3)) = 0$.*

Proof. As a consequence of Theorem 3.2, we see that the center of this algebra is zero.

From the matrix form in Proposition 3.3 $\mathcal{R}_{e_1}, \mathcal{R}_{x_1}, \mathcal{R}_{x_2}, \mathcal{R}_{x_3}$.

$$\begin{cases} \mathcal{R}_{e_1}(e_i) = e_{i+1},\ 1 \le i \le m_1 - 1, & \mathcal{R}_{e_1}(f_i) = f_{i+1},\ 1 \le i \le m_2 - 1, \\ \mathcal{R}_{e_1}(g_i) = g_{i+1},\ 1 \le i \le m_3 - 1, & \mathcal{R}_{e_1}(x_1) = -e_1, \\ \mathcal{R}_{x_1}(e_i) = ie_i,\ 1 \le i \le m_1, & \mathcal{R}_{x_1}(f_i) = (i-1)f_i,\ 2 \le i \le m_2, \\ \mathcal{R}_{x_1}(g_i) = (i-1)g_i,\ 2 \le i \le m_3, & \mathcal{R}_{x_2}(f_i) = f_i,\ 1 \le i \le m_2, \\ \mathcal{R}_{x_3}(f_i) = g_i,\ 1 \le i \le m_3. \end{cases}$$

Thus, we have

$$\mathcal{D}er(\mathcal{R}) = \alpha_1\mathcal{R}_{e_1} + \alpha_2\mathcal{R}_{x_1} + \alpha_3\mathcal{R}_{x_2} + \alpha_4\mathcal{R}_{x_3}.$$

□

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